STATIC OUTPUT FEEDBACK STABILIZATION

A story of people and matrices

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At the University of Firenze, during the '70s of the past century, Roberto Conti (1923-2006) introduced a team of young (at that time) students to mathematical control theory: Giuseppe Anichini, Rosamaria Bianchini, Luciano Pandolfi, Paolo Nistri, Pietro Zecca (and myself)

At the beginning Gianna was not in the team.

Conti's interest was basically focused on finite or infinite dimensional linear systems, but around 1975, Claude Lobry was visiting Firenze, giving some lectures about the new (at that time) topic of geometric nonlinear control theory.

That was the occasion to enroll Gianna in the team.

Later, Sussmann, Jurdjevic, Hermes and many other mathematicians working on this promising subject visited Firenze and delivered lectures.

The focus was especially on controllability, but Conti's legacy included other topics like stability and feedback stabilization.

We investigated possible extension of some linear stabilization techniques to certain class of nonlinear systems. One of the first papers we published on this problem is

A. Andreini, G. Stefani and A.B., *Global stabilizability of homogeneous vector fields of odd degree*, Systems and Control Letters (1988)

STATIC STATE STABILIZATION

Given a time-invariant linear system

$$\dot{x} = Ax + Bu \qquad x \in \mathbf{R}^n,\ u \in \mathbf{R}^m$$

the *static state feedback stabilization* (SSFS) problem consists of find (if it exists) a matrix K such that the closed-loop system

$$\dot{x} = (A - BK)x$$

(obtained by replacing $u = -Kx$) is asymptotically stable at the origin i.e., the matrix $A - BK$ is Hurwitz (all its eigenvalues have a negative real part).

This problem was basically solved at the time our story begins, and many algebraic characterizations of systems admitting a stabilizing static state feedback were known.

THE ARE APPROACH

The characterization we are interested in is the following one:

A linear system is stabilizable by a static state feedback

$$\Longleftrightarrow$$

$\exists$ positive definite, symmetric matrices P and Q solving the algebraic matrix Riccati equation

(ARE) $$A^{\mathbf{t}}P + PA - PBB^{\mathbf{t}}P = -Q$$

Advantages of the ARE:

- It provides an explicit feedback $u = -\gamma B^{\mathbf{t}} P x$ (for any sufficiently large γ)

- It points out a link with the infinite horizon linear quadratic optimization problem

The following two remarks were the starting point of our paper

- The role of homogeneity
- An equivalent reformulation of the ARE

$$\textbf{(C)} \qquad \exists\ P \text{ such that } \ker B^{\mathbf{t}} P \subseteq \{x : x^{\mathbf{t}} P A x < 0\} \cup \{0\}$$

(basically, a CLF condition)

$$\text{ARE} \quad \begin{array}{c}\Longrightarrow \\ \Longleftarrow\end{array} \quad \begin{array}{c}\text{static state feedback} \\ \text{stabilization}\end{array} \quad \begin{array}{c}\Longrightarrow \\ \Longleftarrow\end{array} \quad \textbf{(C)}$$

The "black" implications are easy, the "red" one is more difficult (R. Conti, Academic Press)

Our main result in the 1988 paper was:

Theorem Consider a system

$$\dot{x} = f(x) + Bu$$

where $f(x)$ is a homogeneous vector field of odd degree k. Assume that condition

$$(\mathbf{C}_{NL}) \qquad \ker B^{\mathbf{t}}P \subseteq \{x : x^{\mathbf{t}}Pf(x) < 0\} \cup \{0\}$$

holds for some P. Then for γ large enough, the system is stabilizable by the homogeneous feedback

$$u = -\gamma||x||^{k-1}B^{\mathbf{t}}Px$$

In fact, we can take $\gamma = 1$ w.l.o.g. (replace P by γP)

A long interact ....

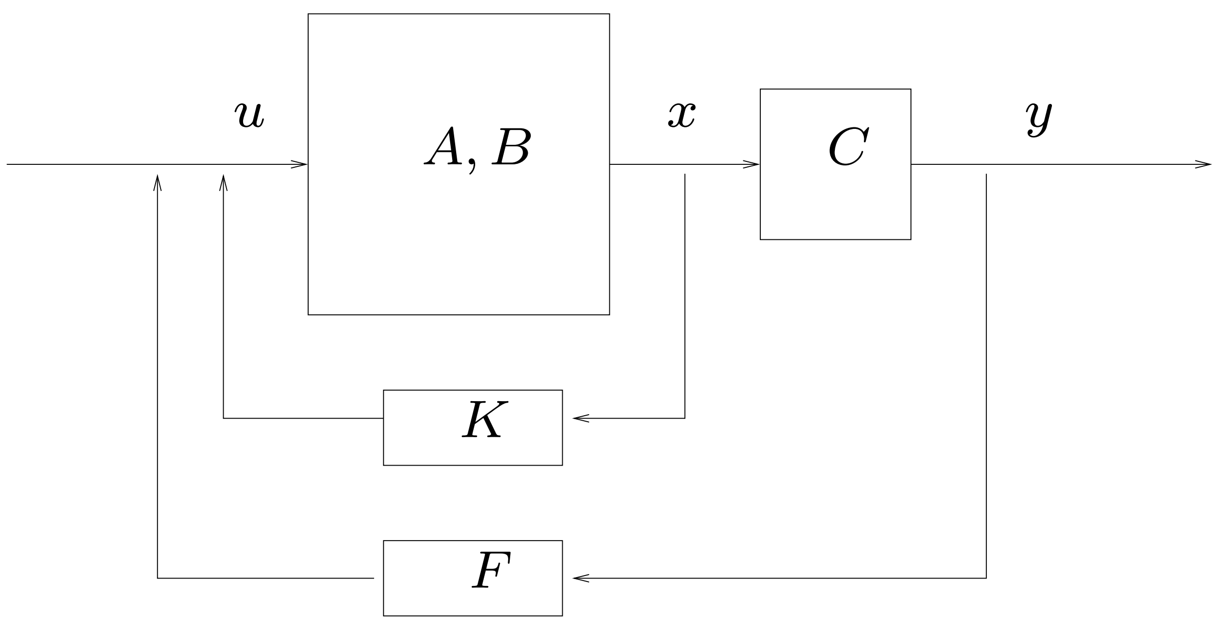
... people take different ways

OUTPUT FEEDBACK STABILIZATION

In view of applications, static feedback of the full state is not feasible in general: only a few of the state variables (or a linear combination of them, $y = Cx$, called the output) can be actually measured and re-injected into the system.

We are so led to the notion of *static output feedback*.

output feedback vs. state feedback



We say that the system (with output map)

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

with $x \in \mathbf{R}^n, \ u \in \mathbf{R}^m, \ y \in \mathbf{R}^p$ is *static output feedback stabilizable* (SOFS) if there exists a matrix F such that the closed-loop system

$$\dot{x} = (A - BFC)x$$

(obtained by replacing $u = -Fy$) is asymptotically stable at the origin.

It turns out that the solution of this problem is much more intriguing.

• There are examples of completely controllable and completely observable linear systems which are not SOFS

• There are systems with the following property: for each stabilizing output feedback $u = -Fy$, we may obtain other stabilizing output feedback of the form $u = -\gamma F y$ but only if $\gamma \in (\gamma_1, \gamma_2)$.

$$A = \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

(stabilizable by $u = -\gamma y$ only if $1 < \gamma < 2$)

A LOOK AT THE LITERATURE

D.S. Bernstein, *Some open problems in matrix theory arising in linear systems and control*, Linear Algebra and its Applications (1992)

"*One of the most basic unsolved problems in control theory is the problem of (static) output feedback stabilizability* "

R. Brockett, *A stabilization problem*, in Open Problems in Mathematical Systems and Control Theory, Springer (1999)

"*It is, of course, well known that the usual output stabilization problem, asking about the existence of a constant feedback gain that stabilizes the system, does not have a especially clean answer.* "

V.D. Blondel, J.N. Tsitsiklis, *A survey of computational complexity results in systems and control*, Automatica (2000)

“*...widely studied and still unsolved, static output feedback problem... a satisfactory answer to this problem has yet to be found. This problem is often cited as one of the difficult open problems in systems and control. Still, despite various attempt, it is unclear whether the problem is NP-hard* ”

In spite of these citations, some attempts to solve the SOFS problem were actually performed during the last decade of the past century: an account can be found in

V.L. Syrmos, C.T. Abdallah, P. Dorato, K. Grigoriadis, *Static output feedback - A survey*, Automatica 1997

STATE OF THE ART

We are in particular interested to possible solutions based on extensions of the ARE and/or the CLF conditions.

We do not follow a chronological development, but rather we present the material according to the approach

THE ARE APPROCH

(assuming rank$C = p$)

The first paper in this direction is:

A. Trofino-Neto, V. Kucera *Stabilization via static output feedback*, IEEE Trans. AC (1993)

It makes use of the notion of Moore-Penrose pseudo-inverse.

Let C be a matrix with p rows and n columns ($p \leq n$). The *Moore-Penrose pseudo-inverse* of C is the (unique) matrix $C^\dagger$ with n rows and p columns such that

$$CC^\dagger C = C \quad \text{and} \quad C^\dagger CC^\dagger = C^\dagger$$

$$CC^\dagger \text{ and } C^\dagger C \text{ are symmetric.}$$

The space $\mathbf{R}^n$ can be decomposed as $\text{im}\,(C^{\mathbf{t}}) \oplus \text{ker}\,(C)$. Moreover, the subspaces $\text{im}\,(C^{\mathbf{t}})$ and $\text{ker}\,(C)$ are orthogonal each other. The matrix $E_{\text{im}} = C^\dagger C$ represents the orthogonal projection on $\text{im}\,(C^{\mathbf{t}})$, while the orthogonal projection on $\text{ker}\,(C)$ is given by $E_{\text{ker}} = I - E_{\text{im}}$.

Trofino-Neto, Kucera Theorem:

The system is stabilizable via static output feedback IFF

(A1) There exist a matrix L, positive definite matrices P, Q and $\gamma > 0$ such that the modified ARE

$$A^{\mathbf{t}}P + PA - \gamma E_{\mathsf{im}}(B^{\mathbf{t}}P + L)^{\mathbf{t}}(B^{\mathbf{t}}P + L)E_{\mathsf{im}} = -Q$$

holds. Moreover, when **(A1)** holds, a stabilizing feedback can be taken of the form $u = -\gamma Fy$, with $F = (B^{\mathbf{t}}P + L)C^{\dagger}$ (again, $\gamma = 1$ w.l.o.g.).

COMMENTS

• The idea of "adding L" is borrowed from KLYP Theory (optimal stabilization w.r.t. quadratic cost, where however, L is known)

• The proof of the implication SOFS $\Longrightarrow$ **(A1)** is correct, but, unfortunately, the reverse implication is false

This was pointed out by a counterexample in

Y.Y. Cao, Y.X. Sun, W.J. Mao, *A new necessary and sufficient condition...*, IEEE Trans. AC 1998

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & -1/2 \end{pmatrix}$$

In the same paper the authors propose the following correction:

(A2) There exist a matrix L, and positive definite matrices P, Q such that

$$A^{\mathbf{t}}P + PA - E_{\mathsf{im}}(B^{\mathbf{t}}P + L)^{\mathbf{t}}(B^{\mathbf{t}}P + L)E_{\mathsf{im}} + S^{\mathbf{t}}S = -Q$$

where $S = LE_{\mathsf{im}} - B^{\mathbf{t}}PE_{\mathsf{ker}}$

While there is no doubt that **(A2)** is sufficient for SOFS, the proof that it is necessary is only sketched, and non convincing.

A convincing necessary and sufficient condition appears in

Y.Y. Cao, J. Lam, Y.X. Sun, *Static output feedback stabilization...*, Automatica 1998

Theorem The system is stabilizable via static output feedback IFF

(A3) There exist a matrix M, and positive definite matrices P, Q such that

$$A^{\mathrm{t}}P + PA - PBB^{\mathrm{t}}P + (B^{\mathrm{t}}P - ME_{\mathsf{im}})^{\mathrm{t}}(B^{\mathrm{t}}P - ME_{\mathsf{im}}) = -Q$$

Moreover, when **(A3)** holds, a stabilizing feedback can be taken of the form $u = -Fy$, with $F = MC^{\dagger}$.

(M and L are related by $L + B^{\mathrm{t}}P = M$)

THE CLF APPROACH

The notion of CLF goes back to

Zvi Artstein , *Stabilization with Relaxed Controls*, Nonlinear Analysis, Theory, Methods and Applications, (1983)

but remained ignored until

Sontag E.D., *A "Universal" Construction of Artstein's Theorem on Nonlinear Stabilization*, Systems and Control Letters, (1989) The CLF approach was used in

J. Tsinias, N. Kalouptsidis, *Output feedback stabilization* IEEE TAC 1990

to provide the following necessary and sufficient condition for SOFS

$\exists P \;:\; \forall y \;\; \exists u = u(y)$ with $u(0) = 0$ such that $\forall x \neq 0$ for which $Cx = y$ one has

$$x^{\mathbf{t}} P A x + x^{\mathbf{t}} P B u(y) < 0$$

(available also in a nonlinear version)

Using the pseudo-inverse formalism, Tsinias-Kalouptsidis condition reads

(B) There exists M and positive definite P, Q such that

$$x^{\mathbf{t}}PAx - x^{\mathbf{t}}PBME_{\mathsf{im}}x = -x^{\mathbf{t}}Qx$$

Remark It can be proved that **(A3)** $\Longleftrightarrow$ **(B)**; the proof that **(B)** $\Longrightarrow$ **(A3)** requires different P, Q

COMMENTS

Contrary to the prevalent opinion in the emerging literature, some mathematical solutions of the SOFS problem did exist, but have been ignored.

However, the relationship among the different approaches is not completely clear

Moreover, these solutions requires a new matrix (L, M, S depending on the notation) which introduces additional complexity from the computational point of view.

A COMPROMISE BETWEEN ARE AND CLS: OUR OLD WAY

This leads to a sufficient (not necessary) condition for SOFS which does not involve the matrix L

This means we are searching stabilizing feedback of the form $F = -\gamma B^{\mathbf{t}} P C^{\dagger} y$ (i.e., $M = B^{\mathbf{t}} P$)

Notation Given a positive definite matrix P, we set

$$\mathcal{Q}_0(x) = x^{\mathrm{t}}(A^{\mathrm{t}}P + PA)x = 2x^{\mathrm{t}}APx$$

$$\mathcal{Q}_1(x) = x^{\mathrm{t}}(E_{\mathrm{im}}\,PBB^{\mathrm{t}}PE_{\mathrm{im}}\,)x = ||B^{\mathrm{t}}PE_{\mathrm{im}}\,x||^2 \geq 0$$

$$\mathcal{Q}_2(x) = x^{\mathrm{t}}(E_{\mathrm{ker}}\,PBB^{\mathrm{t}}PE_{\mathrm{ker}}\,)x = ||B^{\mathrm{t}}PE_{\mathrm{ker}}\,x||^2 \geq 0$$

$$\mathcal{Q}_3(x) = x^{\mathrm{t}}(PBB^{\mathrm{t}}P)x = ||B^{\mathrm{t}}Px||^2 \geq 0$$

and finally

$$\mathcal{Q}(x) = \mathcal{Q}_1(x) - \mathcal{Q}_2(x) + \mathcal{Q}_3(x)$$

Note that:

• The classical ARE is equivalent to: there exists a positive matrix P such that

$$\mathcal{Q}_0(x) - \mathcal{Q}_3(x) < 0 \ , \quad (x \neq 0)$$

• Trofino-Neto, Kucera condition **(A1)** (for $L = 0$) is equivalent to

$$\mathcal{Q}_0(x) - \mathcal{Q}_1(x) < 0 \ , \quad (x \neq 0)$$

• Cao, Sun, Mao condition **(A2)** (for $L = 0$) is equivalent to

$$\mathcal{Q}_0(x) - \mathcal{Q}_1(x) + \mathcal{Q}_2(x) < 0 \ , \quad (x \neq 0)$$

• Cao, Lam, Sun, condition **(A3)** (for $L = 0$) is equivalent to

$$\mathcal{Q}_0(x) + \mathcal{Q}_2(x) - \mathcal{Q}_3(x) < 0 \ , \quad (x \neq 0)$$

• Tsinias-Kalouptsidis condition **(B)** (for $L = 0$) is equivalent to

$$\mathcal{Q}_0(x) - \mathcal{Q}(x) = \mathcal{Q}_0(x) - \mathcal{Q}_1(x) + \mathcal{Q}_2(x) - \mathcal{Q}_3(x) < 0 \ , \quad (x \neq 0)$$

Note that **(A3)** implies **(B)**

Having in mind possible extensions to nonlinear homogeneous systems, it is convenient to reformulate the basic sufficient condition for SOFS as

Lemma Condition

($\mathbf{B}_\gamma$) $\exists P, \gamma \ : \mathcal{Q}_0(x) - \gamma \mathcal{Q}(x) < 0 \ , \quad (x \neq 0)$

implies SOFS with $F = -\gamma B^{\mathbf{t}} P C^{\dagger} y$

Of course, we expect that **($\mathbf{B}_\gamma$)** is not necessary: confirmed by the example

$$A = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Next we provide an equivalent formulation of **($\mathbf{B}_\gamma$)**

Additional notation:

$$\mathbf{S} = \{x \in \mathbf{R}^n : ||x|| = 1\}$$

Moreover, for a given P,

$$\mathbf{S}^+ = \{x \in \mathbf{S} : \mathcal{Q}_0(x) \geq 0\}$$

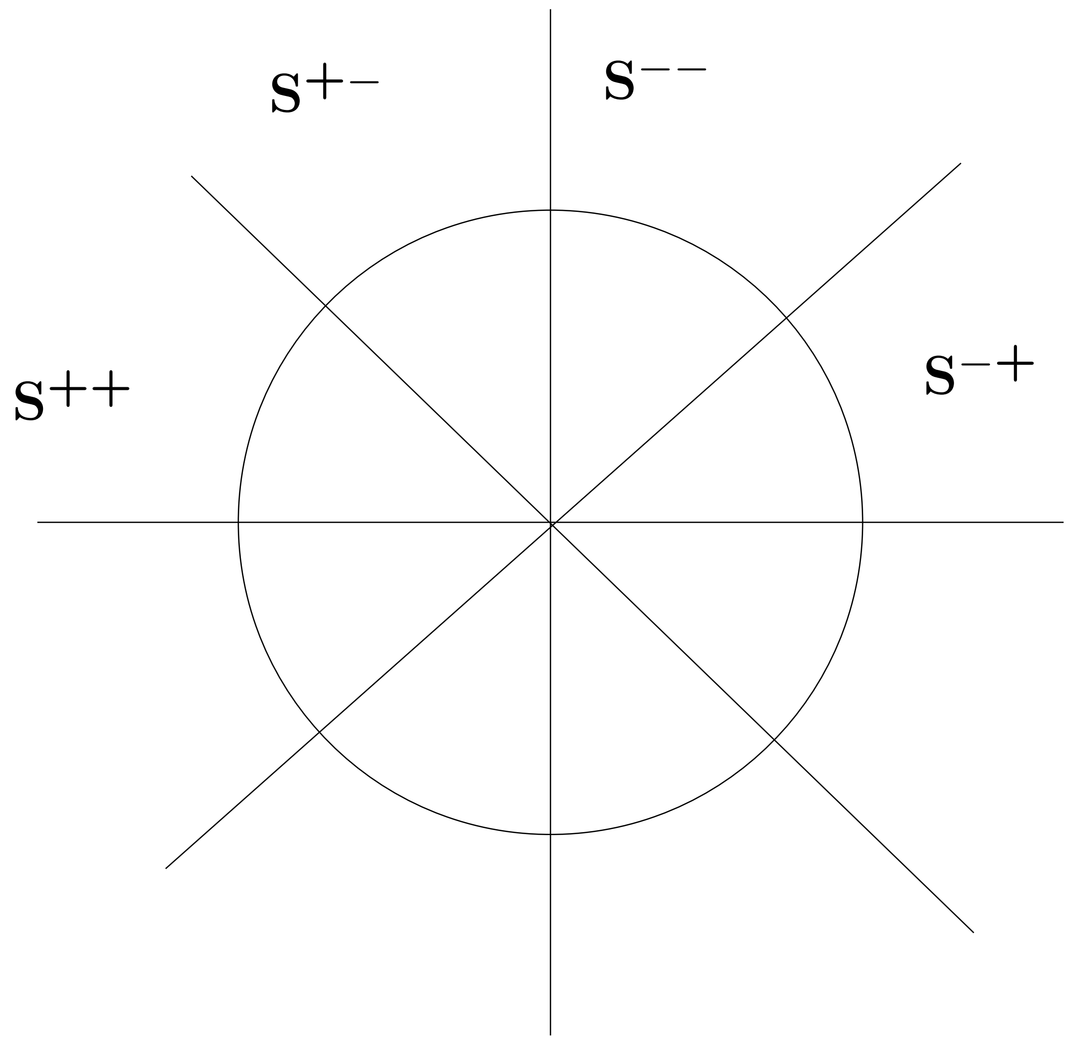
$$\mathbf{S}^- = \{x \in \mathbf{S} : \mathcal{Q}_0(x) < 0\} = \mathbf{S} \setminus \mathbf{S}^+$$

$$\mathbf{S}^{++} = \{x \in \mathbf{S} : \mathcal{Q}_0(x) \geq 0, \mathcal{Q}(x) > 0\}$$

$$\mathbf{S}^{--} = \{x \in \mathbf{S} : \mathcal{Q}_0(x) < 0, \mathcal{Q}(x) < 0\}$$

$$\mathbf{S}^{-+} = \{x \in \mathbf{S} : \mathcal{Q}_0(x) < 0, \mathcal{Q}(x) \geq 0\}$$

$$\mathbf{S}^{+-} = \{x \in \mathbf{S} : \mathcal{Q}_0(x) \geq 0, \mathcal{Q}(x) \leq 0\}$$



Partition of the unit sphere **S**

Theorem Condition **($\mathbf{B}_\gamma$)** holds if and only if

(H1) $x \in \mathbf{S}^+ \Longrightarrow \mathcal{Q}(x) > 0$ (equivalent to $\mathbf{S}^{+-} = \emptyset$)

and

(H2)

$$\sup_{x \in \mathbf{S}^{--}} |\mathcal{Q}(x)| \cdot \sup_{x \in \mathbf{S}^{++}} \mathcal{Q}_0(x) \leq \inf_{x \in \mathbf{S}^{--}} |\mathcal{Q}_0(x)| \cdot \inf_{x \in \mathbf{S}^{++}} \mathcal{Q}(x)$$

Condition **(H2)** identifies the “right” values of γ.

An extension to systems with nonlinear (homogeneous of odd degree k) drift term can be obtained replacing

$$\mathcal{Q}_0(x) \quad \text{by} \quad \mathcal{Q}_{\mathsf{NL}}(x) = 2x^{\mathbf{t}}Pf(x)$$

Theorem If conditions **(H1)** and **(H2)** holds for some P, then there exists a constant $\gamma > 0$ such that the system is stabilizable by the static output feedback

$$u = -\gamma||x||^{k-1}B^{\mathbf{t}}PC^{\dagger}y$$

Finally, we may extend the previous results, removing the restriction $L \neq 0$

Besides the quadratic forms $\mathcal{Q}_0(x)$, $\mathcal{Q}_1(x)$, $\mathcal{Q}_2(x)$, $\mathcal{Q}_3(x)$, we consider

$\mathcal{R}_1(x) = ||(B^{\mathrm{t}}P + LE_{\mathrm{im}})x||^2 \geq 0$

$\mathcal{R}_2(x) = ||LE_{\mathrm{im}}\,x||^2 \geq 0$

and re-define $\mathcal{Q}(x) = \mathcal{Q}_1(x) - \mathcal{Q}_2(x) + \mathcal{R}_1(x) - \mathcal{R}_2(x)$

With this new notation the Lemma and Theorem above remain valid, providing a more general sufficient condition for SOFS, with feedback

$$u = -\gamma||x||^{k-1}(L + B^{\mathrm{t}}P)C^{\dagger}y$$

END OF THE TALK, BUT NOT OF THE STORY...