Inverse optimal control: the sub-Riemannian case

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Mathematical Control Theory,
Porquerolles, June 27-30 2017

Outline

1. Inverse optimal control
2. Inverse sub-Riemannian problem
   - Statement of the problem
   - Affine and projective equivalences
   - Examples
3. Results
   - Existing results
   - Generic metric
   - Nilpotent approximation and generic distribution

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Inverse optimal control

(Direct) Optimal control problem

Given a dynamics $\dot{x} = f(x,u)$ and a cost $C(x_u)$: for any pair of points x_0, x_1, find a trajectory x_{u^*} solutions of

$$\inf\{C(x_u) \ : \ x_u \text{ sol. of } \dot{x} = f(x,u) \text{ s.t. } x_u(0) = x_0, x_u(T) = x_1\}.$$

Inverse optimal control problem

Given $\dot{x} = f(x,u)$ and a set Γ of trajectories: find a cost $C(x_u)$ such that every $\gamma \in \Gamma$ is solution of

$$\inf\{C(x_u) \ : \ x_u \text{ sol. of } \dot{x} = f(x,u) \text{ s.t. } x_u(0) = \gamma(0), x_u(T) = \gamma(T)\}.$$

Applications to analysis/modelling of human motor control (physiology)

$\rightarrow$ looking for optimality principles

Inverse optimal control

Inverse optimal control problem

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$\mathcal{C}$ – family of cost functions

- Optimal control problem: operator $\Phi : C \in \mathcal{C} \mapsto \Gamma$ (=Optimal synthesis).

  to solve the inverse optimal control problem = to find an inverse Φ^{-1}.

  For the well-posedness of the inverse problem:

  1. Φ surjective
  2. Φ injective $\leftarrow$
  3. Φ^{-1} continuous

- Very few general results:
  - Inverse problem of calculus of variations (since 70's)
  - Linear-Quadratic case [Kalmann 64, Nori-Frezza 04]

Outline

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Special class of problems

- Dynamics: control-linear systems on M $\quad$ (M = smooth manifold)

$$\dot{x} = \sum_{i=1}^{m} u_i f_i(x), \qquad \text{i.e.} \qquad \dot{x} \in D_x = \operatorname{span}\{f_1(x), \ldots, f_m(x)\}$$

  - $\operatorname{Lie}_x(f_1, \ldots, f_m) = T_x M$ (LBG)
  - $D = \cup_{x \in M} D_x$ and $D \subset TM$ is a distribution of constant rank

- Class $\mathcal{C}$: Energy functionals or Length functionals associated with positive quadratic forms $g_x(\cdot)$

$$E_g(x_u) = \int_0^T g_{x_u(t)}(\dot{x}_u(t))dt \quad \text{or} \quad L_g(x_u) = \int_0^T \sqrt{g_{x_u(t)}(\dot{x}_u(t))}dt$$

$$(x_u \text{ minimizes } L_g \quad \text{and} \quad g_{x_u}(\dot{x}_u) = \text{const}) \quad \Longleftrightarrow \quad x_u \text{ minimizes } E_g$$

- if $\operatorname{rank} D = \dim M \;\rightarrow\; g$ Riemannian metric on M $\;(D = TM)$
- if $\operatorname{rank} D < \dim M \;\rightarrow\; g$ sub-Riemannian metric on D

Inverse sub-Riemannian (or Riemannian) problem

Let M be a manifold and D a distribution on it.

1. How to recover a metric g from all the minimizers of the energy E_g;
2. How to recover a metric g from all the minimizers of the length L_g.

Injectivity: Can we recover g in a unique way?

unique up to a constant since g and cg have same minimizers

- the set $\{x_u(t) \ : \ x_u(t) \text{ minimizes } E_g\}$ is invariant under affine $t \mapsto at + b$ reparametrization
- the set $\{x_u(t) \ : \ x_u(t) \text{ minimizes } L_g\}$ is invariant under any reparametrization

Outline

Geodesics in sub-Riemannian geometry

Pontryagin Maximum Principle $\Rightarrow$ all minimizers are geodesics

Two kinds of sub-Riemannian geodesics:

- Normal geodesics: Hamiltonian equations, locally minimizing;
- Abnormal geodesics: only determined by D, independent of g.

Definition

Two metrics $g, \widetilde{g}$ on D are:

- **affinely equivalent** if their normal geodesics coincide (up to affine reparametrisation);
- **projectively equivalent** if their normal geodesics coincide as unparameterized curves.

If $g = c\widetilde{g}$ ($c = \text{const}$), then $g, \widetilde{g}$ trivially proj. and aff. equivalent

Equivalence vs inverse sR problem

Theorem (Jean-M-Zelenko, 2016)

For two metrics on D:

- *same energy minimizers* $\Rightarrow$ *affinely equivalent metrics*
- *same shortest paths* $\Rightarrow$ *projectively equivalent metrics*

Definition

A metric g on D is (proj. or aff.) **rigid** if it admits no nontrivial (proj. or aff.) equivalent metric

Consequences of the Theorem:

- all metrics on D are **aff. rigid** $\Rightarrow$ the inverse sR problem with **energy minimizers** is injective
- all metrics on D are **proj. rigid** $\Rightarrow$ the inverse sR problem with **shortest paths** is injective

Outline

Example of nontrivial equivalent metrics

On $M = \mathbb{R}$

- All Riemannian metrics are projectively equivalent.
- Two Riemannian metrics are affinely equivalent $\Leftrightarrow$ constantly proportional

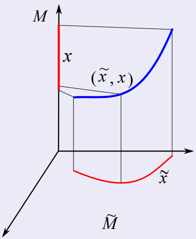
Product space

Let $(M, D, g), (\widetilde{M}, \widetilde{D}, \widetilde{g})$ be sub-Riemannian manifolds.

- $\mathbf{D} = D \oplus \widetilde{D}$ distribution on $\mathbf{M} = M \times \widetilde{M}$
- For $\lambda \in (0, 1)$, $g^\lambda = \lambda g + (1 - \lambda)\widetilde{g}$ metric on $\mathbf{D}$,

$$g^\lambda_{(x,\widetilde{x})}\left(\dot{x}, \dot{\widetilde{x}}\right) = \lambda g_x\left(\dot{x}\right) + (1 - \lambda)\widetilde{g}_{\widetilde{x}}\left(\dot{\widetilde{x}}\right)$$

- $\mathbf{x}(\cdot) = (x, \widetilde{x})(\cdot)$ geodesic of g^λ $\iff$ $x(\cdot)$ geodesic of g and $\widetilde{x}(\cdot)$ geodesic of $\widetilde{g}$,



$\Rightarrow$ For any λ, μ, g^λ and g^μ are (proj. and aff.) equivalent

Levi-Civita pair

$g, \widetilde{g}$ form **Levi-Civita pair** on D near $x_0 \in M$ if on $U =$ neighb. of x_0

- D admits a product structure "$D = D_1 \times \cdots \times D_N$", i.e.

  $U = U_1 \times \cdots \times U_N$ and D_s is a LBG distribution on U_s
  D decomposes in a direct sum $D = D_1 \oplus \cdots \oplus D_N$ where

- $g, \widetilde{g}$ have the form of product metrics:
  for each s, $\exists\, g^s_{x_s}$ Riemannian metric on D_s, x_s – coordinates on U_s, s.t.

$$g_x = \sum_{s=1}^{N} \gamma_s(x) g^s_{x_s} \quad \text{and} \quad \widetilde{g}_x = \sum_{s=1}^{N} \lambda_s(x)\gamma_s(x) g^s_{x_s}$$

with $\lambda_s(x), \gamma_s(x)$ of special form and independent of x_i if $\dim U_i > 1$.

- $g, \widetilde{g}$ are projectively equivalent on U
- $g, \widetilde{g}$ are affinely equivalent on U iff all λ_s, γ_s are constants

Conjecture

Two metrics are projectively equivalent iff they form a Levi-Civita pair, and affinely equivalent iff moreover all λ_s, γ_s constants.

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Existing results

Conjecture is true near generic points:

- in Riemannian case ($D = TM$) [Levi-Civita, 1896], [Eisenhart, 1923]
- in contact and quasi-contact sR cases [Zelenko, 2006]

<u>Remark:</u> Contact distribution doesn't admit a product structure.

Corollary

If D is a contact distribution, then all metrics on D are proj. and aff. rigid

$\rightarrow$ injectivity of the inverse sR problem for a contact distribution

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New results [F. Jean, S.M., I. Zelenko (2017)]

Theorem

D distribution on M — $g, \widetilde{g}$ sR metrics on D

If g and $\widetilde{g}$ are projectively equivalent and non proportional, then their normal geodesic flows admit nontrivial quadratic first-integrals ($\neq$ Hamiltonian).

Generically, g doesn't admit nontrivial quadratic first-integrals

Corollary

On a fixed D

- *a generic metric is affinely rigid*
- *all metrics $\tilde{g}_x$ proj. equivalent to a generic metric g are conformal to g (i.e. $g_x(\cdot) = \alpha^2(x)\tilde{g}_x(\cdot)$)*

Outline

A step toward the conjecture

Theorem

Under the assumptions

- *D is left-invariant distribution on a Carnot group $\mathbb{G}$*
- *$g, \tilde{g}$ are left-invariant metrics on D*

If g and $\tilde{g}$ are projectively equivalent, then ($g, \tilde{g}$ are affinely equivalent), D admits a product structure and $g, \tilde{g}$ form a Levi-Civita pair.

Theorem

$$D \text{ distribution on } M \quad — \quad g, \widetilde{g} \text{ sR metrics on } D$$

If $g, \tilde{g}$ are projectively equivalent et non-proportional, then the nilpotent appr. $\hat{D}$ of D admits a nontrivial product structure and $\widehat{g}, \widehat{\tilde{g}}$ form a Levi-Civita pair.

Corollary (ongoing work)

On a generic D of rank m metrics are affinely rigid and all proj. equivalent metrics are conformal except the cases:
$(\mathrm{rank} D, \dim M) = (m, n) = (2k+1, 2k+2)$ *and* $(m, n) = (4, 6)$, *...*

Some key ideas of the proof

- g_1 and g_2 projectively equivalent $\Rightarrow$ existence of **orbital diffeo.**:

$$\Phi : T^*M \to T^*M \qquad \text{and} \qquad \Phi_* \vec{h}_1 = a\vec{h}_2$$

- in local coordinates on $T^*_x M$, $\Phi = (\Phi_1, \ldots, \Phi_n)$ satisfies

$$A\Phi = b$$

  A is $(\infty \times n)$ matrix

- A^{k+1} is first $(k+1)m$ rows of A and extensions of Jacobi curve J^k are related

$$\text{rank} A^{k+1} = \text{rank} J^k$$

Thank you for your attention!