Second Order Analysis of Control-affine Problems with Control and State Constraints

María Soledad Aronna

Escola de Matemática Aplicada, Fundação Getulio Vargas, Rio de Janeiro, Brazil

Mathematical Control Theory
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-
(joint works with A. Dmitruk, J.-F. Bonnans, B.-S. Goh, P. Lotito)

The problem

$$
\begin{aligned}
\min \quad & J(u) := \varphi(x(0), x(T)) \\
\text{s.t.} \quad & \dot{x}(t) = f(x(t), u(t)) := f_0(x(t)) + \sum_{i=1}^{m} u_i(t)\, f_i(x(t)), \quad \text{a.e. } [0,T], \\
& \Phi(x(0), x(T)) \in K_\Phi = \{0\}_{\mathbb{R}^{n_1}} \times \mathbb{R}_-^{n_2}, \\
& u_{i,\min} \le u(t) \le u_{i,\max}, \quad i = 1, \dots, m, \quad \text{a.e. } [0,T].
\end{aligned}
$$

Control and state spaces:

$$L^\infty(0,T); \quad W^{1,\infty}([0,T]; \mathbb{R}^n).$$

Assuming feasibility, one has existence of optimal trajectory $(\hat{x}, \hat{u})$.

Hamiltonian and Lagrangian functions

Let $\lambda := (\beta, \Psi, p) \in \mathbb{R}_+ \times \mathbb{R}^{(n_1+n_2)*} \times W^{1,\infty}([0,T];\mathbb{R}^{n*})$. Let us define the (unmaximized) pre-Hamiltonian:

$$H[\lambda](x,u,t) := p(t) \sum_{i=0}^{m} u_i f_i(x),$$

the endpoint Lagrangian:

$$\ell^{\beta,\Psi}(x(0),x(T)) := \beta\varphi(x(0),x(T)) + \Psi.\Phi(x(0),x(T)),$$

and the Lagrangian function

$$\mathcal{L}[\lambda](x,u) := \ell^{\beta,\Psi}(x(0),x(T)) + \int_0^T p(t)\Big(f(x(t),u(t)) - \dot{x}(t)\Big)\mathrm{d}t.$$

First order optimality conditions

(I) Costate equation: for $p \in W^{1,\infty}([0,T];\mathbb{R}^{n*})$,

$$-\dot{p}(t) = p(t) f_x(x(t), u(t)), \quad \text{a.e. on } [0,T],$$

with endpoint conditions

$$(-p(0), p(T)) = D\ell^{\beta,\Psi}(x(0), x(T)).$$

(II) Minimization of Hamiltonian (affine w.r.t. u_i) for a.a. $t \in [0,T]$,

$$\begin{cases} u_i(t) = u_{i,\min}, & \text{if } p(t) f_i(x(t)) > 0, \\ u_i(t) = u_{i,\max}, & \text{if } p(t) f_i(x(t)) < 0, \\ p(t) f_i(x(t)) = 0, & \text{if } u_{i,\min} < u_i(t) < u_{i,\max}. \end{cases}$$

First order optimality conditions

(III) Final constraints multiplier: $\Psi \in \mathbb{R}^{(n_1+n_2)*}$;

$$\Psi_i \geq 0, \qquad \Psi_i \Phi_i\big(x(0), x(T)\big) = 0, \quad i = n_1 + 1, \dots, n_2.$$

$\Lambda(x,u) =$ set of Pontryagin multipliers associated to (x,u) :

$$\lambda := (\beta, \Psi, p) \in \mathbb{R}_+ \times \mathbb{R}^{(n_1+n_2)*} \times W^{1,\infty}([0,T];\mathbb{R}^{n*})$$

that satisfy (I)-(III) and $|\beta| + |\Psi| = 1.$

PMP: $\Lambda \neq \emptyset.$

Singular arcs

General definition ([Bryson-Ho 1975]): u has a **singular arc** on $(a, b) \subset [0, T]$ if

(i) Stationarity condition: $H_u = 0$ a.e. on (a, b),

(ii) $H_{uu} = 0$ a.e. on (a, b).

For each component: when $u_{i,\min} < u_i(t) < u_{i,\max}$, on $(a, b) \subset [0, T]$, we say that u_i **has a singular arc** on (a, b).

If $u_i(t) = u_{i,\min}$ on (a, b) or $u_i(t) = u_{i,\max}$ on (a, b) then u_i **has a bang arc** on (a, b).

Bang and singular sets/arcs

Bang sets for the control constraint:

$$B_{i,-} := \{t \in [0,T] : \ \hat{u}_i(t) = u_{i,\min}\},$$
$$B_{i,+} := \{t \in [0,T] : \ \hat{u}_i(t) = u_{i,\max}\},$$

and set $B_i := B_{i,-} \cup B_{i,+}$. And the singular set

$$S_i := \{t \in [0,T] : u_{i,\min} < \hat{u}_i(t) < u_{i,\max}\}.$$

$B, S-$arcs are *maximal* intervals contained in these sets.

Previous second order conditions

Necessary conditions: **Kelley, Kopp, Moyer, Goh, Gabasov, Kirillova, Krener, Levitin, Milyutin, Osmolovskii, Agrachev, Sachkov, Gamkredlidze, more recently, Frankowska & Tonon,** others...

Sufficient conditions: **Moyer, Dmitruk, Sarychev, Poggiolini & Stefani, Zezza, Bonnard, Caillau & Trélat, Maurer, Osmolovskii; Sussmann, Schättler, Jankovic,** others...

Assumptions on control structure & Complementarity

(i) For each i, the interval $[0, T]$ is (up to a zero measure set) the disjoint union of finitely many arcs of type B and S,

(ii) the control $\hat{u}_i$ is at uniformly positive distance of the bounds, over S arcs,

(iii) strict complementarity for the control constraints:
$B_{i,+} = \{t \in [0, T] : \max_{\lambda \in \Lambda} H_{u_i}[\lambda] < 0\}$,
$B_{i,-} = \{t \in [0, T] : \min_{\lambda \in \Lambda} H_{u_i}[\lambda] > 0\}$.

These hypotheses are verified in many practical examples: see e.g. aerospace applications by Bonnans, Martinon, Trélat, Maurer; Goh's fishing problem, others aerospace applications; Ledzewicz & Schättler's chemotherapy optimization problems, many other examples...

Jumping may be a necessary condition:
[McDanell-Powers, *Necessary conditions for joining optimal singular and nonsingular arcs*, SIAM J. Control, 1971]

For example: the Goddard problem in 1D (vertical motion of a rocket)

[Goddard, 1919] [Seywald & Cliff, 1993]

State variables: (h, v, m) are the position, speed and mass.

Control variable: u is the normalized thrust (proportion of $\mathcal{T}_{\max}$)

Cost: final mass of the rocket (or minimize fuel consumption)

$$
\begin{aligned}
&\max\ m(T),\\
&\text{s.t.}\ \ \dot{h}(t) = v(t),\\
&\qquad \dot{v}(t) = -1/h(t)^2 + 1/m(t)\big(\mathcal{T}_{\max}\, u(t) - D(h(t), v(t))\big),\\
&\qquad \dot{m}(t) = -b\,\mathcal{T}_{\mathsf{max}}\, u(t),\\
&\qquad 0 \le u(t) \le 1, \quad \text{a.e. on } [0, T],\\
&\qquad h(0) = 0,\ v(0) = 0,\ m(0) = 1,\ h(T) = 1,
\end{aligned}
$$

T : free final time, b : fuel consumption coefficient,
$\mathcal{T}_{\mathsf{max}}$ is the maximal thrust, $D(h, v)$ is the drag

Numerical solution for Goddard problem

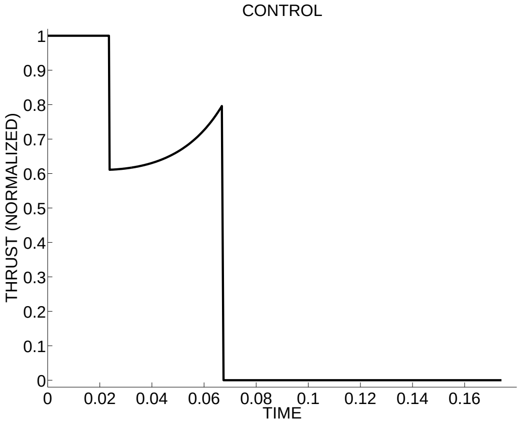

Figure: Goddard Problem

Critical directions

$z[v, z^0]$ solution of *linearized state equation* for $(v, z^0) \in L^2 \times \mathbb{R}^n$

Tangent cone to the endpoint constraints:

$T_\Phi := \{0\}_{\mathbb{R}^{n_1}} \times \{\eta \in \mathbb{R}^{n_2} : \eta_i \leq 0$ if $\Phi_{n_1+i}(\hat{x}(0), \hat{x}(T)) = 0,\ 1 \leq i \leq n_2\}$.

The critical cone:

$$\mathcal{C} := \left\{ \begin{array}{l} \big(z[v, z^0], v\big) \in W^{1,\infty} \times L^\infty : \varphi'(\hat{x}(0), \hat{x}(T))(z(0), z(T)) \leq 0 \\ \Phi'(\hat{x}(0), \hat{x}(T))(z(0), z(T)) \in T_\Phi,\ \ v_i = 0 \text{ a.e. on } B_i,\ \forall i \end{array} \right\}.$$

Second order necessary optimality condition

Weak minimum: exists $\varepsilon > 0$ such that
$\varphi(\hat{x}(0), \hat{x}(T)) \leq \varphi(x(0), x(T))$ for any feasible (x, u) for which

$$\|(x, u) - (\hat{x}, \hat{u})\|_\infty < \varepsilon$$

For $\lambda \in \Lambda$, we set

$$Q[\lambda](z, v) := D^2_{(x,u)^2} \mathcal{L}[\lambda](\hat{x}, \hat{u})(z, v)^2,$$

for $(z = z[v, z_0], v) \in W^{1,\infty} \times L^\infty$, then

$$Q[\lambda](z, v) = D^2 \ell^{\beta,\Psi}(z(0), z(T))^2 + \int_0^T \left(z^\top H_{xx}\, z + 2v^\top H_{ux}\, z\right) \mathrm{d}t.$$

Theorem (Second order necessary condition)

Assume that $(\hat{x}, \hat{u})$ is a weak minimum. Then

$$\max_{\lambda \in \Lambda} Q[\lambda](z, v) \geq 0, \qquad \textit{for all } (z, v) \in \mathcal{C}.$$

Q does not contain a term in H_{uu}!

Second order necessary optimality condition

Weak minimum: exists $\varepsilon > 0$ such that
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Q does not contain a term in H_{uu}!

Goh's Transformation [Goh 1966]

$$(z, v) \quad \mapsto \quad \left(y := \int v,\ \xi := z - f_u y \right).$$

$$\dot{z} = f_x\, z + f_u\, v \quad \mapsto \quad \dot{\xi} = f_x\, \xi + By,$$

where $B := f_x f_u - \frac{\mathrm{d}}{\mathrm{d}t} f_u$.

$$\Omega \ \mapsto \ \Omega_{\mathcal{P}}(\xi, y, v) := \tfrac{1}{2} g(\xi(0), \xi(T), y(T)) + \tfrac{1}{2} \int_0^T (\xi^\top H_{xx} \xi + 2y^\top M \xi + y^\top R y + 2v^\top V y) \mathrm{d}t,$$

where

$$M := f_u^\top H_x x - \frac{\mathrm{d}}{\mathrm{d}t} H_{ux} - H_{ux} f_x,\ \ R := f_u^\top H_{xx} f_u - H_{ux} B - (H_{ux} B)^\top + \frac{\mathrm{d}}{\mathrm{d}t} (H_{ux} f_u + (H_{ux} f_u)^\top),$$

$$g(\zeta_0, \zeta_T, h) := D^2 \ell(\zeta_0, \zeta_T + f_u(T) h)^2 + h^\top H_{ux}(T)(2\zeta_T + f_u(T) h),$$

$$V := \tfrac{1}{2}(H_{ux} f_u - (H_{ux} f_u)^\top), \quad V_{ij} = \tfrac{1}{2} p[f_i, f_j].$$

Goh conditions

$$G(\operatorname{co}\Lambda) := \{\lambda \in \operatorname{co}\Lambda : V_{ij}[\lambda] = 0 \text{ on } S_i \cap S_j, \text{ for any } 1 \leq i < j \leq m\}.$$

Theorem

If $(\hat{x}, \hat{u})$ is a weak minimum then

$$\max_{\lambda \in G(\operatorname{co}\Lambda)} \Omega_{\mathcal{P}}[\lambda](\xi, y, v) \geq 0, \quad \text{for } (z, v) \in \mathcal{C}.$$

Corollary (Goh condition)

If $(\hat{x}, \hat{u})$ is a weak minimum then $G(\operatorname{co}\Lambda) \neq \emptyset$ and, for all $\lambda \in G(\operatorname{co}\Lambda)$,

$$p[f_i, f_j] = 0, \quad \textit{on } S_i \cap S_j.$$

Removing v

$$\mathcal{C} \overset{Goh}{\longmapsto} \mathcal{P} := \left\{ \begin{array}{l} (\xi, y, y(T)) : y_i' = 0 \text{ over } B_i,\ \dot{\xi} = f_x \xi + By, \\ \varphi_i'(x(0), x(T))(\xi(0), \xi(T) + f_u(T)h) \leq 0, \\ \eta_j'(x(0), x(T))(\xi(0), \xi(T) + f_u(T)h) = 0 \end{array} \right\}.$$

$$\mathcal{P}_2 := \text{closure of } \mathcal{P} \text{ in } W^{1,2} \times L^2 \times \mathbb{R}^m$$

For $\lambda \in G(\Lambda)$, set

$$\Omega_{\mathcal{P}_2}[\lambda](\xi, y, h) := \Omega_{\mathcal{P}}[\lambda](\xi, y, h) + \Xi(\xi, y, h)$$

Theorem (Second order necessary condition in new variables)

Let $(\hat{x}, \hat{u})$ be a weak minimum, then

$$\max_{\lambda \in G(\operatorname{co} \Lambda)} \Omega_{\mathcal{P}_2}[\lambda](\xi, y, h) \geq 0, \quad \text{for all } (\xi, y, h) \in \mathcal{P}_2.$$

Sufficient condition: scalar control case

Pontryagin minimum: for any $N > 0$, $\exists\ \varepsilon_N > 0$ such that $(\hat{x}, \hat{u})$ is optimal on the set

$$\left\{(x,u) \text{ feasible} : \|(x,u) - (\hat{x},\hat{u})\|_\infty < \varepsilon_N, \quad \|u - \hat{u}\|_1 < \varepsilon_N\right\}.$$

Convergence in the Pontryagin sense: $\|v_k\|_1 \to 0$, $\|v_k\|_\infty < N$.

$\gamma-$order: $(\xi_0, y, h) \in \mathbb{R}^n \times L^2 \times \mathbb{R}$, let

$$\gamma(\xi_0, y, h) := |\xi_0|^2 + \int_0^T y(t)^2 \mathrm{d}t + |h|^2.$$

$\gamma-$growth condition in the Pontryagin sense: exists $\rho > 0$, for every $v_k \to 0$ in the Pontryagin sense,

$$J(\hat{u} + v_k) - J(\hat{u}) \geq \rho\gamma(y_k, y_k(T)),$$

where $y_k := \int v_k$.

Sufficient condition

Theorem (Characterization of quadratic growth)

Suppose that there exists $\rho > 0$ such that

$$\max_{\lambda \in \Lambda} \Omega_{\mathcal{P}_2}[\lambda](\xi, y, h) \geq \rho \gamma(\xi(0), y, h), \quad \text{for all } (\xi, y, h) \in \mathcal{P}_2.$$

Then $(\hat{x}, \hat{u})$ is a Pontryagin minimum satisfying $\gamma-$growth.
Furthermore, if $(\hat{x}, \hat{u})$ is normal, the converse holds.

Remark

In case the bang arcs are absent, this theorem reduces to one proved in [Dmitruk, 1977].

1 Problems with control bounds

2 Problems with control bounds & state constraint

Control bounds & state constraint: the problem

We consider the optimal control problem:

$$\begin{aligned}
&\min \ \varphi(x(0), x(T)) \\
&\qquad \dot{x}(t) = f(u(t), x(t)) = f_0(x(t)) + u(t) f_1(x(t)), \quad \text{a.e. on } [0, T], \\
&\qquad \Phi(x(0), x(T)) \in K_\Phi = \{0\}_{\mathbb{R}^{n_1}} \times \mathbb{R}_-^{n_2}, \\
&\qquad u_{\min} \le u(t) \le u_{\max}, \quad \text{a.e. on } [0, T], \\
&\qquad g(x(t)) \le 0, \ \ t \in [0, T].
\end{aligned}$$

where $g : \mathbb{R}^n \to \mathbb{R}$.

Assuming feasibility, one has existence of optimal trajectory $(\hat{x}, \hat{u})$.

Some applications

Some control-affine problems with control bounds and state constraints:

J.B. Keller, *Optimal velocity in a race*, Amer. Math. Monthly 81 (1974), 474–480.

A.V. Dmitruk, I.A. Samylovskiy: *A simple Trolley-like model in the presence of a nonlinear friction and a bounded fuel expenditure*, Disc. Math., Diff. Inclusions, Control and Optim. 33 (2013), 135–147.

A. Aftalion and J.F. Bonnans, *Optimization of running strategies based on anaerobic energy and variations of velocity*, SIAM J. Applied Math., 2014

P.-A. Bliman, M.S.A., F.C. Coelho, and M.A. da Silva, *Global stabilizing feedback law for a problem of biological control of mosquito-borne diseases,* Proceedings of the 54th IEEE CDC, 2015 & *arXiv preprint March 2015*

Hamiltonian and Lagrangian functions

As before, we have the unmaximized Hamiltonian H and the endpoint Lagrangian ℓ.

Let

$$\lambda := (\beta, \Psi, p, \mathrm{d}\mu) \in \mathbb{R}_+ \times \mathbb{R}^{(n_1+n_2)*} \times BV^{n*} \times \mathcal{M},$$

and define the Lagrangian function

$$\mathcal{L}[\lambda](u,x) := \ell^{\beta,\Psi}(x(0), x(T)) + \int_0^T p\big(f(x,u) - \dot{x}\big)\mathrm{d}t + \int_{[0,T]} g(x)\mathrm{d}\mu(t).$$

First order optimality conditions

(I) Costate equation: for $p \in BV^{n*}$,

$$-\mathrm{d}p(t) = p(t) f_x(u(t), x(t))\mathrm{d}t + g'(x(t))\,\mathrm{d}\mu(t), \quad \text{for a.a. } t \in [0, T],$$

with endpoint conditions

$$(-p(0), p(T)) = D\ell^{\beta,\Psi}(x(0), x(T)).$$

(II) Minimization of Hamiltonian (affine w.r.t. u) for a.a. $t \in [0, T]$,

$$\begin{cases} u(t) = u_{\min}, & \text{if } p(t) f_1(x(t)) > 0, \\ u(t) = u_{\max}, & \text{if } p(t) f_1(x(t)) < 0, \\ p(t) f_1(x(t)) = 0, & \text{if } u_{\min} < u(t) < u_{\max}. \end{cases}$$

First order optimality conditions

(III) Final constraints multiplier: $\Psi \in \mathbb{R}^{(n_1+n_2)*}$;

$$\Psi_i \geq 0, \qquad \Psi_i \Phi_i(x(0), x(T)) = 0, \quad i = n_1 + 1, \dots, n_2.$$

(IV) State constraint multiplier: $\mathrm{d}\mu \in \mathcal{M}$;

$$\mathrm{d}\mu \geq 0; \qquad \int_0^T g(x(t))\mathrm{d}\mu(t) = 0.$$

$\Lambda(x,u) =$ set of Pontryagin multipliers associated to (x,u) :
$\lambda := (\beta, \Psi, p, \mathrm{d}\mu) \in \mathbb{R}_+ \times \mathbb{R}^{(n_1+n_2)*} \times BV^{n*} \times \mathcal{M}$ that satisfy
(I)-(IV) and $|\beta| + |\Psi| = 1$.

PMP: $\Lambda \neq \emptyset$.

Bang, contact and singular arcs

Bang sets for the control constraint:

$$\begin{cases} B_- := \{t \in [0,T] : \ \hat{u}(t) = u_{\min}\}, \\ B_+ := \{t \in [0,T] : \ \hat{u}(t) = u_{\max}\}, \end{cases}$$

and set $B := B_- \cup B_+$.
Contact set associated with the state constraint:

$$C := \{t \in [0,T] : \ g(\hat{x}(t)) = 0\},$$

Singular set

$$S := \{t \in [0,T] : u_{\min} < \hat{u}(t) < u_{\max} \text{ and } g(\hat{x}(t)) < 0\}.$$

$C, B, S-$arcs are maximal intervals contained in these sets.

Assumptions on control structure & Complementarity

(i) the interval $[0, T]$ is (up to a zero measure set) the disjoint union of finitely many arcs of type B, C and S,

(ii) the control $\hat{u}$ is at uniformly positive distance of the bounds, over C and S arcs,

(iii) the control $\hat{u}$ is discontinuous at CS and SC junctions,

(iv) complementarity of control and state constraints, i.e., the corresponding multipliers do not vanish on the active arcs.

The state constraint is of first order, i.e.

$$g'(\hat{x}(t))f_1(\hat{x}(t)) \neq 0, \qquad \text{on } C.$$

These hypotheses are verified in some practical examples.

Regulator problem with state constraint

$$\min \ \tfrac{1}{2}\int_0^5 (x_1^2 + x_2^2)\mathrm{d}t;$$

$$\text{s.t. } \dot{x}_1 = x_2; \quad \dot{x}_2 = u; \quad x_1(0) = 0, \ x_2(0) = 1.$$

subject to the control bounds and state constraint:

$$-1 \leq u(t) \leq 1, \quad x_2 \geq -0.2,$$

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Numerical solution

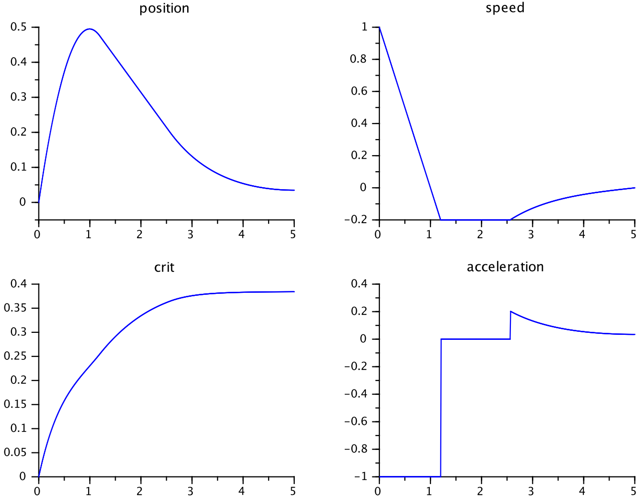

Figure: x_1 : position, x_2 : speed, x_3 : cost, u : acceleration

Second order necessary conditions

The critical cone:

$$\mathcal{C} := \left\{ \begin{array}{l} (z[v, z_0], v) \in W^{1,\infty} \times L^\infty : \varphi'(\hat{x}(0), \hat{x}(T))(z(0), z(T)) \leq 0 \\ \Phi'(\hat{x}(0), \hat{x}(T))(z(0), z(T)) \in T_\Phi, \;\; v(t) = 0 \text{ a.e. on } B \\ g'(\hat{x}(t))z(t) = 0 \;\; \text{on } C \end{array} \right\}.$$

For every multiplier $\lambda \in \Lambda$, we have

$$D^2_{(x,u)^2} \mathcal{L}[\lambda](\hat{u}, \hat{x})(z, v)^2 = Q[\lambda](z, v), \qquad \text{for } (z[v, z_0], v) \in W^{1,\infty} \times L^\infty,$$

where

$$Q := D^2 \ell^{\beta,\Psi}(z(0), z(T))^2 + \int_0^T \left(z^\top H_{xx}\, z + 2v H_{ux}\, z\right) \mathrm{d}t + \int_{[0,T]} z^\top g''\, z \mathrm{d}\mu(t).$$

Theorem (Second order necessary condition)

Assume that $(\hat{x}, \hat{u})$ is a weak minimum. Then

$$\max_{\lambda \in \Lambda} Q[\lambda](v, z) \geq 0, \qquad \textit{for all } (z, v) \in \mathcal{C}.$$

Goh's transform

$$(z, v) \quad \mapsto \quad \left(y := \int v,\ \xi := z - f_u y \right).$$

$$\dot{z} = f_x\, z + f_u\, v \quad \mapsto \quad \dot{\xi} = f_x\, \xi + By,$$

Write $\xi = \xi[y, \xi_0]$.

Set of (strict) primitive critical directions:

$$\mathcal{P} := \left\{ \begin{array}{l} (\xi, y, h) \in W^{1,\infty} \times \mathbb{R} \times W^{1,\infty} : \\ y(0) = 0,\ y(T) = h,\ (\dot{y}, \xi + y f_1) \in \mathcal{C} \end{array} \right\}.$$

Closure of $\mathcal{P}$ in $H^1 \times L^2 \times \mathbb{R}$

Proposition

$(\xi, y, h) \in \overline{\mathcal{P}}$ *iff*

$$\begin{cases} g'(\hat{x}(t))(\xi(t) + y(t) f_1(\hat{x}(t))) = 0 \text{ on } C, \\ y \text{ is constant on each } B \text{ arc}, \\ \phi'(\hat{x}(0), \hat{x}(T))\big(\xi(0), \xi(T) + h f_1(\hat{x}(T))\big) \leq 0, \\ \Phi'(\hat{x}(0), \hat{x}(T))\big(\xi(0), \xi(T) + h f_1(\hat{x}(T))\big) \in T_\Phi, \\ \ y \text{ is continuous at the } BC, CB \text{ and } BB \text{ junctions}, \\ \ y(t) = 0, \text{ on } B_{0\pm} \text{ if a } B_{0\pm} \text{ arc exists}, \\ \ y(t) = h, \text{ on } B_{T\pm} \text{ if a } B_{T\pm} \text{ arc exists}, \\ \ \lim_{t \uparrow T} y(t) = h, \text{ if } T \in C. \end{cases}$$

Set $\mathcal{P}^2 := \overline{\mathcal{P}}$.

Goh transformation on the Hessian of Lagrangian

Set for $(\xi, y, h) \in H^1 \times L^2 \times \mathbb{R}$ and $\lambda := (\beta, \Psi, p, \mathrm{d}\mu) \in \Lambda$,

$$\Omega := \Omega_T + \Omega^0 + \Omega^E + \Omega^g,$$

where

$$\begin{aligned}
\Omega_T[\lambda](\xi, y, h) &:= 2h\, H_{ux}(T)\xi(T) + h\, H_{ux}(T) f_1(\hat{x}(T))h, \\
\Omega^0[\lambda](\xi, y, h) &:= \int_0^T \big(\xi^\top H_{xx}\xi + 2yM\xi + yRy\big)\mathrm{d}t, \\
\Omega^E[\lambda](\xi, y, h) &:= D^2\ell^{\beta,\Psi}(\xi(0), \xi(T) + f_1(\hat{x}(T))h)^2, \\
\Omega^g[\lambda](\xi, y, h) &:= \int_0^T (\xi + f_1(\hat{x})y)^\top g''(\hat{x})(\xi + f_1(\hat{x})y)\mathrm{d}\mu(t).
\end{aligned}$$

Second order necessary condition after Goh transformation

Let $(z[v, z_0], v) \in H^1 \times L^2$ and let $(\xi[y, z_0], y)$ be defined by Goh transformation. Then, for any $\lambda \in \Lambda$,

$$Q[\lambda](z, v) = \Omega[\lambda](y, y_T, \xi).$$

Theorem (Second order necessary condition)

If $(\hat{x}, \hat{u})$ *is a weak minimum, then*

$$\max_{\lambda \in \Lambda} \Omega[\lambda](\xi, y, h) \geq 0, \quad \textit{for all } (\xi, y, h) \in \mathcal{P}^2.$$

Extended critical cone

Define $\mathcal{P}_*^2 \supseteq \mathcal{P}^2$ consisting of $(\xi, y, h) \in H^1 \times L^2 \times \mathbb{R}^m$ verifying

- $g'(\hat{x}(t))(\xi(t) + y(t) f_1(\hat{x}(t))) = 0$ on C,
- y is constant on each B arc,
- $\phi'(\hat{x}(0), \hat{x}(T))\big(\xi(0), \xi(T) + h f_1(\hat{x}(T))\big) \leq 0$,
- $\Phi'(\hat{x}(0), \hat{x}(T))\big(\xi(0), \xi(T) + h f_1(\hat{x}(T))\big) \in T_\Phi$,
- ~~y is continuous at the BC, CB and BB junctions,~~
- $y(t) = 0$, on $B_{0\pm}$ if a $B_{0\pm}$ arc exists,
- $y(t) = h$, on $B_{T\pm}$ if a $B_{T\pm}$ arc exists,
- ~~$\lim_{t \uparrow T} y_t = h$, if $T \in C$~~ (if $T \in C$, $[\mu(T)] = 0$ for all μ)

Second order sufficient condition

Theorem

Suppose that for each $\lambda \in \Lambda$, $\Omega[\lambda](\cdot)$ is a Legendre form in $\{(\xi = \xi[y], y, h) \in H^1 \times L^2 \times \mathbb{R}\}$ and there exists $\rho > 0$ such that

$$\max_{\lambda \in \Lambda} \Omega[\lambda](\xi, y, h) \geq \rho\gamma(\xi_0, y, h), \quad \textit{for all } (\xi, y, h) \in \mathcal{P}_*^2.$$

Then $(\hat{x}, \hat{u})$ is a Pontryagin minimum satisfying $\gamma-$growth.

Remark: (ii) holds if and only if, for every $\lambda \in \Lambda$,

$$R(\hat{x}(t)) + f_1(\hat{x}(t))^\top g''(\hat{x}(t)) f_1(\hat{x}(t)) \nu(t) > \alpha > 0, \quad \text{on } [0, T],$$

where ν is the density of $\mathrm{d}\mu$ [Hestenes 1951].

Applications

- *Application I:* analytical proof of local optimality in some examples (see references)
- *Application II:* convergence of an associated shooting algorithm
- *Application III:* stability under data perturbation

References

M.S.A., J.F. Bonnans, A.V. Dmitruk & P.A. Lotito, *Quadratic order conditions for bang-singular extremals,* Numerical Algebra, Control Optim., AIMS Journal, 2012

M.S.A., J.F. Bonnans & P. Martinon, *A shooting algorithm for optimal control problems with singular arcs,* J. Optim. Theory App., 2013

M.S.A., J.F. Bonnans & B.S. Goh, *Second order analysis of control-affine problems with scalar state constraint.* Math. Programming, 2016

Open problems

- Optimal control continuous at the junction times. For instance:

  B.S. Goh, G. Leitmann, T.L. Vincent, *Optimal control of a prey-predator system*, Math. Biosci. 1974.

- "No-gap" conditions for the single-input state-constrained case? Note that one may have $\mathcal{P}_*^2 \supsetneq \mathcal{P}_S^2$.
- Sufficient condition for the vector control case
- Many, many others..

THANK YOU FOR YOUR ATTENTION