Local study of sub-Finslerian geometry for maximum norms in dimension 2 and 3

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Sub-Finslerian structures we consider

Consider in a neighborhood of $0 \in \mathbb{R}^2$ or $\mathbb{R}^3$ the dynamics

$$\dot{q} = u_1 F_1(q) + u_2 F_2(q), \qquad \max\{|u_1|, |u_2|\} \leq 1$$

The minimizers are understood as minimizers of the time to join two given points.

State of the art

Related works in sub-Riemannian geometry : Local studies of SR metrics, motivated in particular by the link between this local geometry and analysis of the heat kernel.

Local studies : A. Agrachev, B. Bonnard, J.-P. Gauthier and others (since 90's).

Asymptotics of the heat kernel in SR geometry : G. Ben Arous and R. Léandre (80's-90's).

Link between the singularity of exponential map and asymptotics of the heat kernel in SR geometry : D. Barilari, U. Boscain, R. Neel (2012) and others.

State of the art

Few works in sub-Finslerian geometry :

Smooth norms (outside 0) : Jeanne Clelland and Christopher Moseley (2006, 2007).

The sphere of the left invariant sub-Finslerian structure on Heisenberg associated to a maximum norm : E. Breuillard and E. Le Donne (2013).

Extremals (and their numbers of switches before losing optimality) in the case of Grushin, Heisenberg and Martinet : D. Barilari, U. Boscain, E. Le Donne, and M. Sigalotti (2016).

Local existence of minimizers

In all the following we assume that $\mathrm{Lie}_0(\Delta) = T_0\mathbb{R}^n$.

$\rightarrow$ local controllability (Chow-Rashevski)

$\rightarrow$ existence of local minimizers (Filippov)

In the two cases, there is no abnormal. Hence the Hamiltonian of the PMP is

$$\begin{aligned} H(q,\lambda,u) &= u_1\lambda.F_1(q) + u_2\lambda.F_2 - 1 \\ &= u_1\phi_1 + u_2\phi_2 - 1 \end{aligned}$$

where the $\phi_i = \lambda.F_i(q)$ are called the switching functions.

Bang and singular extremal

If $\phi_i(t) > 0$ (resp < 0) then $u_i(t) = 1$ (resp -1) (PMP)

If ϕ_i changes sign at time t then the control $u_i : \pm 1 \to \mp 1$ at t.

Definition

A bang extremal has constant controls with value 1 or -1,
a bang-bang extremal is a concatenation of bangs,
a u_i-singular extremal corresponds to a null switching function ϕ_i,
t is a switching time if u is not bang in any neighborhood of t.

Define $F_3 = [F_1, F_2]$ and $\phi_3 = \lambda.F_3$.

It satisfies $\dot{\phi}_1 = -u_2\phi_3$ and $\dot{\phi}_2 = u_1\phi_3$.

If (F_1, F_2) is a basis of the tangent space, one can define f_1 and f_2 by $F_3 = f_2 F_1 - f_1 F_2$

Among the local extremals that switch more than once, we can identify two types :

- those with two controls switching in short time, corresponding to a large initial covector λ, this case being possible only if

$$\text{span}\{F_1, F_2\} \neq TM.$$

- those with only one control switching at least twice in short time, let say u_1, then ϕ_1 should start with values close to 0. Moreover, since $\dot{\phi}_1 = -u_2\phi_3$, also ϕ_3 should be close to 0.

Define $F_4 = [F_1, [F_1, F_2]]$, $F_5 = [F_2, [F_1, F_2]]$, $\phi_4 = \lambda.F_4$ and $\phi_5 = \lambda.F_5$.

It satisfies $\dot{\phi}_3 = u_1\phi_4 + u_2\phi_5$.

If (F_1, F_2, F_3) is a basis of the tangent space, one can define 6 functions f_{ij} by

$$F_4 = f_{41}F_1 + f_{42}F_2 + f_{43}F_3, \quad F_5 = f_{51}F_1 + f_{52}F_2 + f_{53}F_3,$$

hence

$$\dot{\phi}_3 = u_1(f_{41}\phi_1 + f_{42}\phi_2 + f_{43}\phi_3) + u_2(f_{51}\phi_1 + f_{52}\phi_2 + f_{53}\phi_3).$$

And one can conclude that along a u_1-singular ($\phi_1 \equiv 0$ and $\phi_3 \equiv 0$ and $|\phi_2| \equiv 1$)

$$u_1 f_{42} + u_2 f_{52} \equiv 0.$$

When $n = 2$ we studied such structures in the generic cases that is for a residual set of families of vector fields for the C^{∞}-Whitney topology.

An important tool to establish generic properties of families of vector fields is Thom transversality theorem and corollaries.

Case $n = 2$

It looks quite simple but the zoology is quite rich

$$\begin{aligned} \Delta_A &= \{q \mid F_1(q) /\!\!/ F_2(q)\}, \\ \Delta_1 &= \{q \mid F_1(q) /\!\!/ [F_1, F_2](q)\}, \\ \Delta_2 &= \{q \mid F_2(q) /\!\!/ [F_1, F_2](q)\}. \end{aligned}$$

The sets Δ_A, Δ_1, Δ_2 are generically smooth 1-dimensional submanifolds, transversal by pair.

Abnormal extremals support is included in $\Delta_A \cap \Delta_1 \cap \Delta_2$ which is generically a discrete set, hence, generically, there is no non trivial abnormal.

Theorem

Generically, at each point, one of the three following situations occures

(C1) F_1 *and* F_2 *are not parallel*

(C2) F_1 *and* F_2 *are parallel and* $span(F_1, F_2) \pitchfork T\Delta_A$

(C3) F_1, F_2 *and* $[F_1, F_2]$ *are in* $T\Delta_A$ *but* $[F_1, [F_1, F_2]]$ *is not.*

The same holds if we replace the vectors by

$$G_1 = F_1 + F_2 \quad \text{and} \quad G_2 = F_1 - F_2$$

which are natural to be considered since they are the velocities of bang-bang extremals.

Dimension 2 : normal form

The first case (C1) holds true outside Δ_A. We define locally a normal coordinate system (x, y) by fixing

- $G_2 = \partial_y$ along the y-axis
- $G_1 = \partial_x$.

In this system,

$$G_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} x g_1(x,y) \\ 1 + x g_2(x,y) \end{pmatrix}$$

where g_1 and g_2 are smooth functions.

(C1) corresponds to Riemannian points in almost Riemannian geometry

The second case (C2) holds true along Δ_A except at isolated points. We define coordinates in such a way that

$$G_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} g_3(x,y) \\ x g_4(x,y) \end{pmatrix}, \quad \Delta_A = \{x = 0\}$$

where g_3 and g_4 are smooth functions such that $g_4(0,y) = 1$ (condition fixing the parameterization of Δ_A).

(C2) corresponds to Grushin points in almost Riemannian geometry

Finally, the case (C3) holds true at isolated points of Δ_A and we build coordinates such that

$$G_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} g_5(x,y) \\ g_6(x,y) \end{pmatrix}, \quad \Delta_A = \{ay = -\frac{1}{2}x^2 + o(x^2)\}$$

with $\frac{\partial^2 g_5}{\partial x^2}(0,y) = 0$ and
$g_6(x,y) = ay + \frac{1}{2}x^2 + by^2 + cxy + O_3(x,y)$ where $a \neq 0$.

(C3) corresponds to tangency points in almost Riemannian geometry

Case $n = 3$, contact distribution

Contact : $\text{span}\{F_1, F_2, F_3 = [F_1, F_2]\} = \mathbb{R}^3$

$\rightarrow$ No abnormal extremal.

Generically, in dimension 3, the set of points where the distribution is contact has for complement a surface called the Martinet surface.

In our work we restricted our study by adding non nullity conditions on certain invariants appearing in the jets of the couple (F_1, F_2) hence we consider points outside a finite union of surfaces.

Dimension 3, contact case : normal form

We consider again $G_1 = F_1 + F_2$ and $G_2 = F_1 - F_2$ and define coordinates (x, y, z) such that we get the normal form

$$G_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} xg_7(x,y,z) \\ 1 + xg_8(x,y,z) \\ xg_9(x,y,z) \end{pmatrix}$$

where $g_7(0,0,z) = 0$, $g_8(0,0,z) = 0$ and $g_9(0,0,z) = 1$.

Outside Δ_A, one computes easily that if $\phi_1(t) = 0$ then

$$\dot{\phi}_1(t) = f_1(q(t))u_2(t)\phi_2(t) = f_1(q(t)).$$

Hence, outside $\Delta_A \cup \Delta_1$, the only possible switch for u_1 is

- $-1 \to +1$ if $f_1(q(t)) > 0$,
- $+1 \to -1$ if $f_1(q(t)) < 0$.

The same holds for u_2 and f_2.

The support of u_i-singular extremals is entirely included in Δ_i.

Moreover, along a piece of Δ_1 that does not intersect $\Delta_A \cup \Delta_2$, if

- G_1 and G_2 point on the same side,
- G_1 points in the domain where $f_1 > 0$,

then a singular optimal extremal run on it. It is then called a u_1-turnpike.
If one of the two conditions does not hold then no extremal can run on it.

For Δ_2, the conditions are

- G_1 and $-G_2$ point on the same side,
- G_1 points in the domain where $f_2 > 0$.

see U. Boscain et al, 2005.

Dim 2 : properties of the minimizers

Proposition (Cut)

- *In the cases (C2) and (C3), the cut locus is not empty.*
- *If* $q \notin \Delta_A \cup \Delta_1 \cup \Delta_2$ *then its local cut locus is empty.*
- *If* $q \in (\Delta_1 \cup \Delta_2) \setminus \Delta_A$*, the existence of local cut locus depends on invariants appearing in the normal form.*

Dim 2 : examples

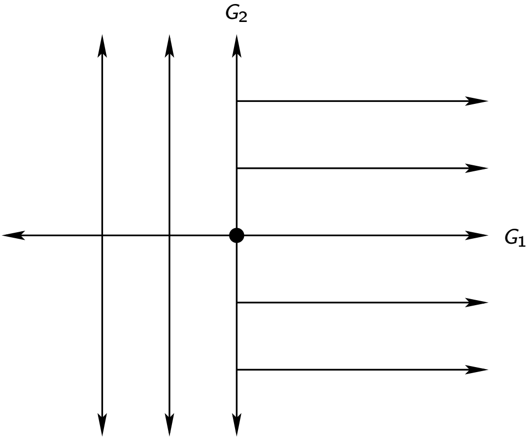


Figure – $f_1 > 0$ and $f_2 > 0$

Dim 2 : examples

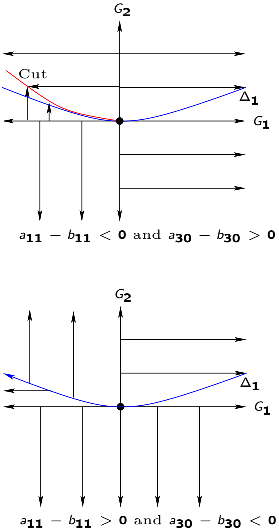


Figure – $f_1 = 0$, $f_2 > 0$ and G_1 tangent to Δ_1

Dim 2 : examples

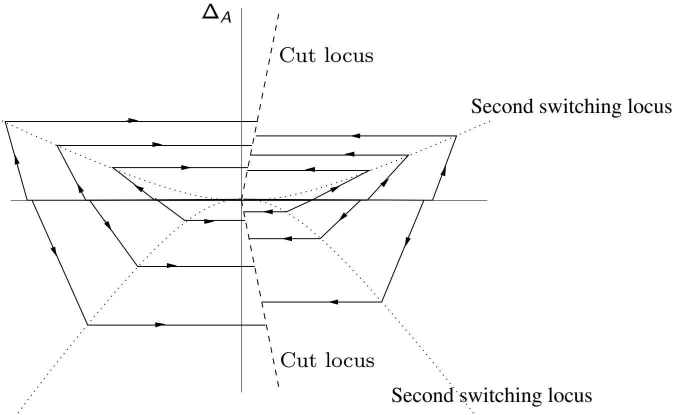


Figure – $q \in \Delta_A \setminus (\Delta_2 \cup \Delta_1)$

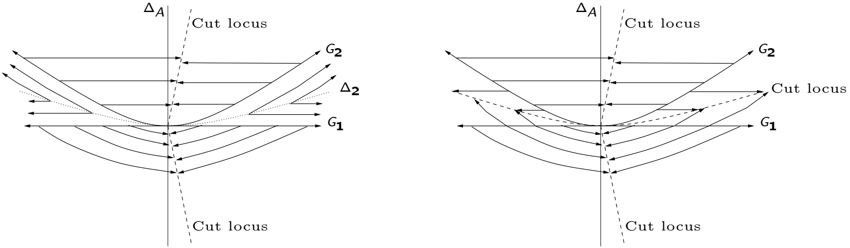


Figure – $q \in (\Delta_A \cap \Delta_2) \setminus \Delta_1$

Dim 2 : examples

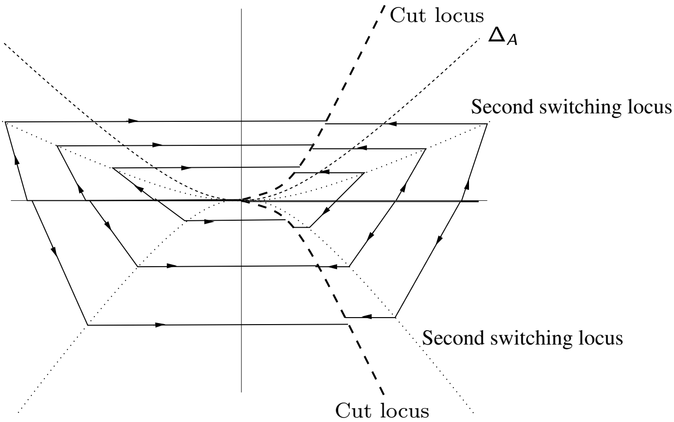


Figure – $q \in \Delta_A \cap \Delta_1 \cap \Delta_2$

The nilpotent approximation is given by

$$\widehat{G_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \widehat{G_2} = \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix}$$

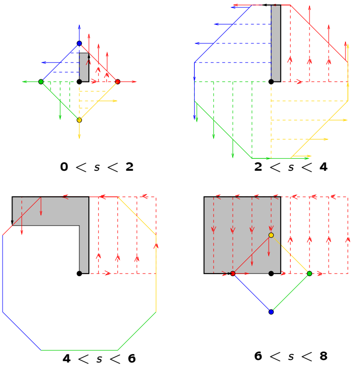


Figure – Evolution of the front at $\lambda_z \neq 0$ fixed. In red dot lines and in black the extremals with initial speed G_1, in full line the front at 4 different times, with four colors corresponding to the four possible initial speeds

Figure – Three points of view of the part of the sphere generated by extremals with both controls switching, in the nilpotent case

We first concentrate our attention on extremals with both controls switching in short time. They have $|\lambda_z| \gg 1$.

They are all bang-bang.

Following the techniques used in the 3d-contact case in sub-Riemannian geometry (see Agrachev et al), one can make the change of coordinates

$$r = \frac{1}{\lambda_z}, \quad s = \frac{t}{r}, \quad p_x = r\lambda_x, \quad p_y = r\lambda_y.$$

Dim 3 : extremals with u_1 and u_2 switching

Denoting $p = (p_x, p_y, 1)$ and $q = (x, y, z)$ one gets the equations for the extremals

$$
\begin{aligned}
\frac{dq}{ds} &= r(u_1 F_1 + u_2 F_2) \\
\frac{dp}{ds} &= r(-p(u_1 dF_1 + u_2 dF_2) + (p(u_1 \frac{\partial F_1}{\partial z} + u_2 \frac{\partial F_2}{\partial z}))p) \\
\frac{dr}{ds} &= r^2 p(u_1 \frac{\partial F_1}{\partial z} + u_2 \frac{\partial F_2}{\partial z})
\end{aligned}
$$

These equations are integrable for u constant. Hence, computing also the switching functions, one can compute the jets with respect to $r(0)$ of bang-bang extremals.

Dim 3 : extremals with u_1 and u_2 switching

Corresponding to these extremals, we can define an exponential map.

During the two first bangs of an extremal, the jacobian of the exponential is null but is not during the third and fourth bangs.

We define the conjugate time as the infimum of times t such that the jacobian changes sign before t.

We call conjugate point the corresponding point.

Dim 3 : extremals with u_1 and u_2 switching

Proposition (Conjugate and cut times)

The conjugate times T_{conj} of bang-bang extremals starting with velocity $\pm G_1$ and $r(0)$ small enough are all equal to the fourth switching time T_4, or all equal to the fifth switching time T_5. The same hold for $\pm G_2$.

The cut time T_{cut} is smaller than T_{conj}.

If $T_{conj} = T_4$, then $T_3 \leq T_{cut} \leq T_4$.

If $T_{conj} = T_5$, then $T_4 \leq T_{cut} \leq T_5$.

Dim 3 : extremals with u_1 and u_2 switching

If $T_{conj} = T_4$ for all the extremals with r_0 small then the picture of the closure of the front is given by

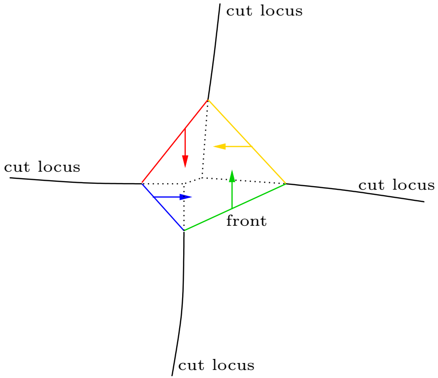
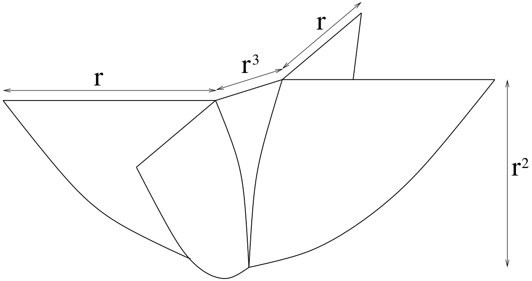
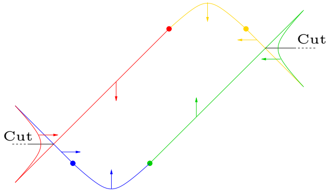


Figure – Closure of the cut locus at z fixed.

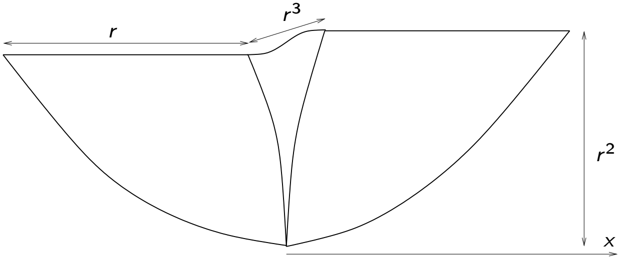
and the upper part of the cut locus is



If $T_{conj} = T_4$ for the extremals starting with $\pm G_1$ and $T_{conj} = T_5$ for the extremal starting with $\pm G_2$ then the picture of the closure of the front is given by

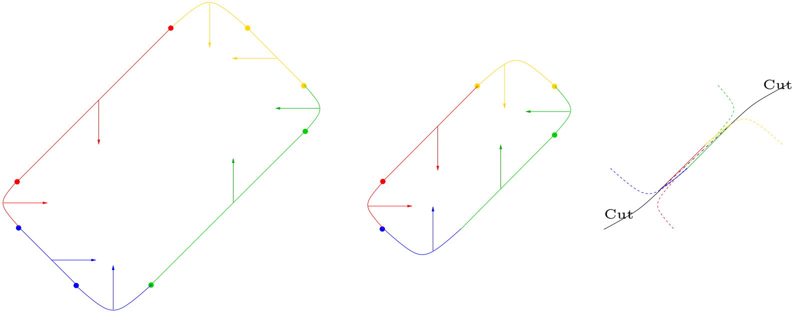


and the picture of the upper part of the cut locus is

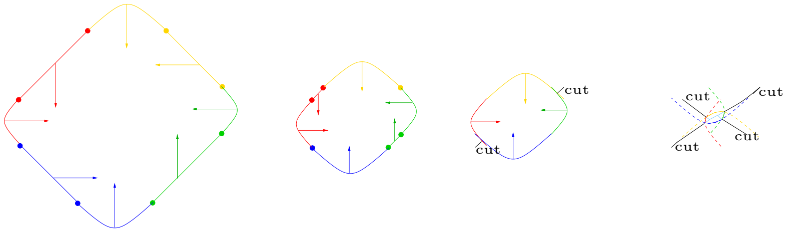


Dim 3 : extremals with u_1 and u_2 switching

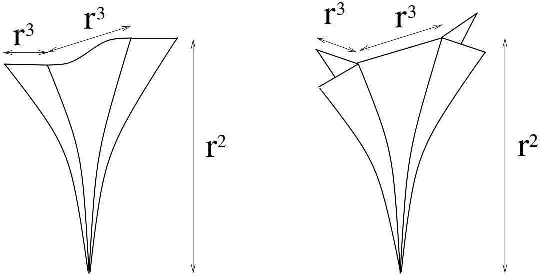
If $T_{conj} = T_5$ for all the extremal, then the closure of the front is like



or like



and the possible pictures of the upper part of the cut locus are



One compute easily that, if $u_1 \sim 1$ and u_2 switches at least twice in short time,

$$\ddot{\phi}_2(t) \sim (f_{41} + u_2 f_{51}).$$

Depending on the relative values of f_{41} and f_{51}, the possible evolutions of the function ϕ_2 are shown in the following pictures.

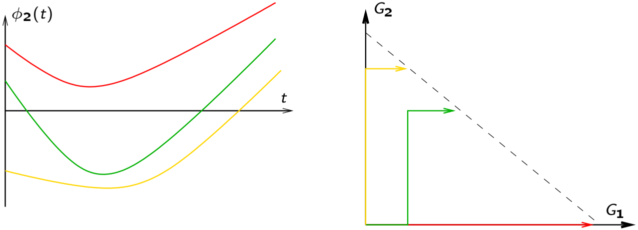


Figure – Extremals when $|f_{51}| < f_{41}$

$$\ddot{\phi}_2(t) = (f_{41} + u_2 f_{51}) \;\Rightarrow\; \phi_2 \text{ convex}$$

No cut.

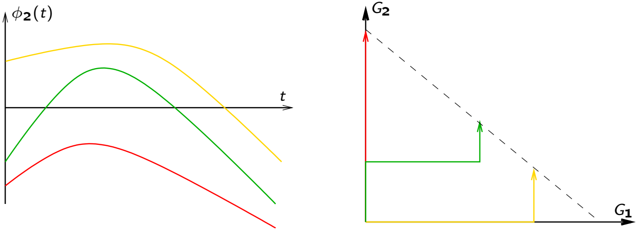


Figure – Extremals when $|f_{51}| < -f_{41}$

$$\ddot{\phi}_2(t) = (f_{41} + u_2 f_{51}) \;\Rightarrow\; \phi_2 \text{ concave}$$

No cut.

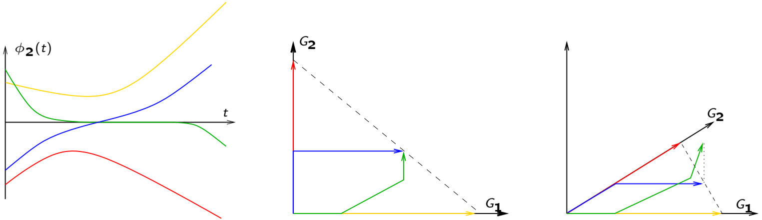


Figure – Extremals when $|f_{41}| < f_{51}$

$$\ddot{\phi}_2(t) = (f_{41} + u_2 f_{51}) \Rightarrow sign(\ddot{\phi}_2) = sign(\phi_2)$$

The green curve has a singular piece.

No cut.

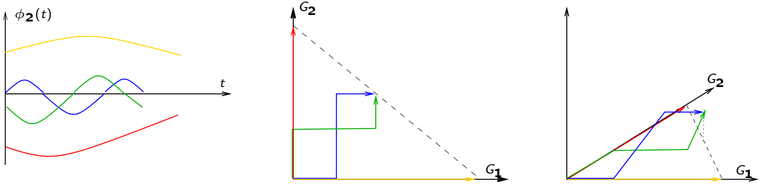


Figure – Extremals when $|f_{41}| < -f_{51}$

$$\ddot{\phi}_2(t) = (f_{41} + u_2 f_{51}) \Rightarrow sign(\ddot{\phi}_2) = -sign(\phi_2)$$

Presence of cut.

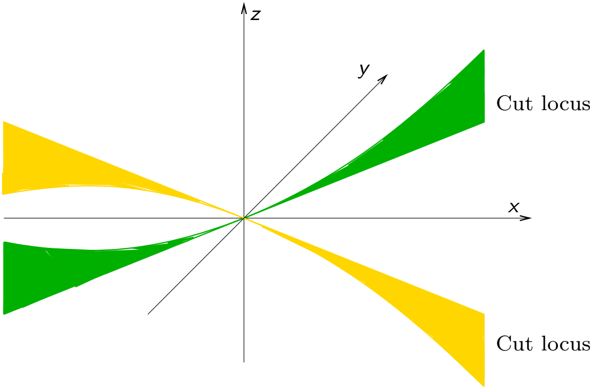


Figure – Part of the cut locus generated by the extremal with one control switching at least twice when $|f_{41}| < -f_{51}$ and $|f_{52}| < f_{42}$

What about stability ?