Mathematical Control Theory,
with a special session in honor of Gianna Stefani
Porquerolles, France, June 27th-30th 2017

A Hamiltonian approach to sufficiency in OC

Parametric LQ Problems

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IBM Selectric typewriter

Figure : A technological breakthrough

"Golf Balls"

Figure : Changing fonts

A. A. Agrachev, G. Stefani, and P. Z.
An invariant second variation in optimal control.
Internat. J. Control, 71(5):689–715, 1998.

A. A. Agrachev, G. Stefani, and P. Z.
Strong minima in optimal control.
Tr. Mat. Inst. Steklova, 220(Optim. Upr., Differ. Uravn. i Gladk. Optim.):8–26, 1998.

A. A. Agrachev, G. Stefani, and P. Z.
A Hamiltonian approach to strong minima in optimal control.
In *Differential geometry and control (Boulder, CO, 1997)*, volume 64 of *Proc. Sympos. Pure Math.*, pages 11–22. Amer. Math. Soc., Providence, RI, 1999.

A. A. Agrachev, G. Stefani, and P. Z.
Strong optimality for a bang-bang trajectory.
SIAM J. Control Optim., 41(4):991–1014 (electronic), 2002.

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- To use the symplectic properties of the cotangent bundle to compare the costs of neighbouring admissible trajectories by lifting them to the cotangent bundle.

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- To use the symplectic properties of the cotangent bundle to compare the costs of neighbouring admissible trajectories by lifting them to the cotangent bundle.
- To define in the cotangent bundle T^*M a suitable Hamiltonian flow $\mathscr{H}_t$ emanating from a horizontal Lagrangian submanifold Λ. This flow is the one of the maximised Hamiltonian when this last is at least C^2.

- To estimate the variation of the cost of admissible trajectories by the variation of a function of their final points and, if it is the case, their final times.

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- To obtain a suitable second order approximation (2^{nd} *variation*) in the form of a coordinate-free Linear-Quadratic (LQ) problem and to require its coercivity.

- To estimate the variation of the cost of admissible trajectories by the variation of a function of their final points and, if it is the case, their final times.
- To obtain a suitable second order approximation (2^{nd} *variation*) in the form of a coordinate-free Linear-Quadratic (LQ) problem and to require its coercivity.
- To show that $\mathscr{H}_{t*}$ (the derivative of $\mathscr{H}_t$ along the reference extremal) is, up to an isomorphism, the linear Hamiltonian flow associated to the LQ problem.

- To use the coercivity of the 2^{nd} variation to substitute the manifold described by the transversality conditions, Λ_0, by an horizontal one Λ. This can be obtained by adding a penalty term which reduces the problem to a problem with free initial point and whose 2^{nd} variation is still coercive.

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- To deduce for a problem with free initial point and fixed final point that the projection on M of $\mathscr{H}_t$ emanating from Λ is locally invertible so that we can go back to the first issue and we can compare the costs of neighbouring admissible trajectories by lifting them to the cotangent bundle.

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- To deduce for a problem with free initial point and fixed final point that the projection on M of $\mathscr{H}_t$ emanating from Λ is locally invertible so that we can go back to the first issue and we can compare the costs of neighbouring admissible trajectories by lifting them to the cotangent bundle.
- To use again the coercivity of the 2^{nd} variation to complete the proof for the general case.

G. Stefani, and P. Z.
A Hamiltonian approach to sufficiency in optimal control with minimal regularity conditions: Part I
in Variational Methods in Imaging and Geometric Control, 496–516, De Gruyter, 2017

A Non-Linear OC Problem

- M is a finite dimensional manifold
- $X_1, \ldots, X_m$ are smooth vector fields on M
- $[0, T] \in \mathbb{R}$
- $x_0 \in M$
- $\Delta = \{u = (u_1, .., u_m) \in \mathbb{R}^m \colon u_i \geq 0, i = 1, \ldots, m, \sum_{i=1}^m u_i = 1\}$

A Non-Linear OC Problem

An OC problem (fixed-free)

$$\textbf{Minimize} \quad c(\xi(T))$$

subject to

$$\dot{\xi}(t) = \sum_{i=1}^{m} \upsilon(t) X_i(\xi(t)), \qquad \text{a.e. } t \in [0, T]$$

$$\upsilon \in L^{\infty}([a, b], \Delta)$$

$$\xi(0) = x_0, \quad \xi(T) \in M$$

A Non-Linear OC Problem

We assume there exist times $\widehat{\tau}_1, \widehat{\tau}_2$, $0 =: \widehat{\tau}_0 < \widehat{\tau}_1 < \widehat{\tau}_2 < \widehat{\tau}_3 := b$, vector fields h_1, h_2, $h_3 \in \{X_1, \dots X_m\}$, and a measurable function $\widehat{\upsilon} \in L^\infty([\widehat{\tau}_2, T], (0,1))$ such that the solution $\widehat{\xi}$ to

bang-bang-singular

$$\begin{aligned}
\dot{\xi}(t) &= h_1(\xi(t)) && t \in (0, \widehat{\tau}_1) \\
\dot{\xi}(t) &= h_2(\xi(t)) && t \in (\widehat{\tau}_1, \widehat{\tau}_2) \\
\dot{\xi}(t) &= \widehat{\upsilon}(t) h_3(\xi(t)) + (1 - \widehat{\upsilon}(t))\, h_2(\xi(t)) && t \in (\widehat{\tau}_2, T)
\end{aligned}$$

is admissible and it satisfies Pontryagin Maximum Principle (PMP).

The Extended Second Variation

The extended second variation is a L-Q problem on $[\widehat{\tau}_2, T]$.

The Extended Second Variation

$$J_{\mathrm{ext}}[(\varepsilon_0, \delta x, \varepsilon_1, \upsilon)]^2 =$$

$$= \frac{\varepsilon_0^2}{2} \left(L_k^2 c\,(\widehat{x}_b) + H_{12}(\widehat{\ell}_1) \right) - \varepsilon_1 L_{\delta x} L_{f_1} c\,(\widehat{x}_b) - \frac{\varepsilon_1^2}{2} L_{f_1}^2 c\,(\widehat{x}_b) +$$

$$+ \frac{1}{2} \int_a^b \left\{ 2\upsilon(t) L_{\zeta(t)} L_{\dot{g}_t} c\,(\widehat{x}_b) + \upsilon(t)^2 R(t) \right\} \mathrm{d}t$$

The Extended Second Variation

subject to

The control equation

$$\dot{\zeta}(t) = \upsilon(t)\,\dot{g}_t(\widehat{x}_b)$$

with end-point conditions

$$\zeta(a) = \varepsilon_0\,k(\widehat{x}_b), \quad \zeta(b) = \delta x + \varepsilon_1 f_1(\widehat{x}_b)$$

A source of inspiration

G. STEFANI AND P. Z., *Constrained Regular LQ-Control Problems*, SIAM J. Control 1997

Parametric L-Q problems

On a given compact interval $[a, b] \subseteq \mathbb{R}$ we consider the following parametric linear control system

The control equation

$$\dot{\xi}(t) = B(t)\upsilon(t), \; t \in [a, b]$$
$$\xi(a) = N_a \, \alpha \quad \text{and} \quad \xi(b) = N_b \, \alpha,$$

with $\alpha \in \mathbb{R}^k$

Parametric L-Q problems

with associated quadratic form

The quadratic cost

$$J[(\alpha, \upsilon)]^2 := \frac{1}{2}\Gamma[\alpha]^2 + \frac{1}{2}\int_a^b \left\{2\langle Q(s)\xi(s), \upsilon(s)\rangle + R(s)[\upsilon(s)]^2\right\} ds$$

Parametric L-Q problems

The control system plays the role or a constraint and hence the quadratic form J is defined on

Admissible couples

$$\mathscr{V} := \Big\{(\alpha, v) \in \mathbb{R}^k \times L^2\Big([a,b], \mathbb{R}^m\Big) \mid \text{the system has a solution}\Big\}$$

Parametric L-Q problems

It is easy to see that $(\alpha, v) \in \mathscr{V}$ if and only if

Admissibility

$$\int_a^b B(s)v(s)ds + (N_a - N_b)\,\alpha = 0,$$

Parametric L-Q problems

therefore if we define

$$\Sigma : (\alpha,\, \upsilon) \mapsto \int_a^b B(s)\upsilon(s)ds + (N_a - N_b)\,\alpha,$$

Then the domain of J is the Hilbert space

The Hilbert Space

$$\mathscr{V} = \ker \Sigma$$

Strengthened Legendre-Clebsch Condition

Our main assumption is the Strengthened Legendre-Clebsch Condition (SLCC)

Assumption

$$\textit{there is } h_0 > 0 \textit{ such that } \langle R(t)u,\, u\rangle \geq h_0\,||u||^2,\ \ t \in [a,\, b]$$

Preliminary Analysis

Different values of the parameter α might yield a zero boundary conditions on both end-points

$$W_0 := \ker N_a \cap \ker N_b$$

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and if we take the zero control then the couples in

$$\mathscr{W}_0 = \{(\alpha, 0) \mid \alpha \in W_0\}$$

are admissible since $\Sigma(\alpha, 0) = 0$ for $(\alpha, 0) \in \mathscr{W}_0$

A Necessary Condition

A first necessary condition for $J > 0$ ($J \geq 0$) is

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$$J_{|\mathscr{W}_0} > 0 \ \left(J_{|\mathscr{W}_0} \geq 0\right)$$

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A Necessary Condition

$$J_{|\mathscr{W}_0} > 0 \ \left(J_{|\mathscr{W}_0} \geq 0\right) \text{ iff } \Gamma_{|W_0} > 0 \ \left(\Gamma_{|W_0} \geq 0\right)$$

The finite dimensional part

Denote by

$$Z_0 := \ker\left(\Gamma_{|W_0}\right) = \{\alpha \in W_0 \mid \alpha \text{ is } \Gamma\text{-orthogonal to } W_0\}$$

Z_0 is the subspace of the Γ-transversal of W_0.

• The elements in $(\alpha, 0), \alpha \in Z_0$ are couples (parameter, zero control) that not only generate a zero trajectory but, moreover, they have zero integral cost and zero parameter cost.

Preliminary Analysis

Decompose W_0 as

$$W_0 = Z_0 \oplus Z_1$$

where Z_0 and Z_1 are mutually Γ-orthogonal and $\Gamma_{|Z_0} = 0$, $\Gamma_{|Z_1} > 0$.

Preliminary Analysis

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- For J to be positive we must have that

A Necessary Condition for $J > 0$

$$Z_0 = \{0\}$$

Preliminary Analysis

We can decompose $\mathbb{R}^k$ as

$$\mathbb{R}^k = W_0 \oplus W_1$$

and they are Γ-orthogonal to each other and if we define

$$\mathscr{W}_1 = \{(\alpha, v) \in \mathscr{V} \mid \alpha \in W_1\}$$

Decoupling

$$\mathscr{V} = \mathscr{W}_0 \oplus \mathscr{W}_1$$

The Decoupling

For

$$(\alpha,\, v) = (\alpha_0,\, 0) + (\alpha_1, v), \text{ with } (\alpha_0,\, 0) \in \mathscr{W}_0,\; (\alpha_1, v) \in \mathscr{W}_1\,.$$

The Decoupling

$$J[(\alpha,\, v)]^2 = \frac{1}{2}\Gamma[\alpha_0]^2 + J[(\alpha_1, v)]^2,\; \alpha_0 \in W_0,\, \alpha_1 \in W_1.$$

As a consequence $J_{|\mathscr{W}_0} = \Gamma_{|W_0}$ is a finite dimensional form which can be decoupled and studied separately.

Decoupling

The finite dimensional part can be examined separately hence we assume that

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Assumption

$$W_0 := \ker N_a \cap \ker N_b = \{0\}$$

and hence

$$\mathscr{V} = \mathscr{W}_1 .$$

Preliminary Analysis

If we denote by

$$N : \mathbb{R}^k \to \mathbb{R}^{2n},\ \alpha \mapsto \begin{pmatrix} N_a\,\alpha \\ N_b\,\alpha \end{pmatrix}$$

if $W_0 = \{0\}$ then N admits a left inverse $N^\sharp$ hence

$$N\alpha = \begin{pmatrix} \zeta(a) \\ \zeta(b) \end{pmatrix}$$

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Boundary Conditions and the Parameter α

$$\begin{pmatrix} \zeta(a) \\ \zeta(b) \end{pmatrix} \in \operatorname{Im} N \quad \text{with} \quad \alpha = N^\sharp \begin{pmatrix} \zeta(a) \\ \zeta(b) \end{pmatrix}.$$

A Necessary Condition

There could also be constant solutions

$$\mathscr{V}_0 := \{(\alpha, 0) \mid N_a \alpha = N_b \alpha\} \simeq V_0 := \ker(N_a - N_b).$$

A second necessary condition for $J > 0$ ($J \geq 0$) is

A Necessary Condition

$$J_{|\mathscr{V}_0} > 0 \ \left(J_{|\mathscr{V}_0} \geq 0\right)$$

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A family of subspaces

For $c \in [0, 1]$ we consider a continuous family of closed nested subspaces $\mathscr{V}_c \subseteq \mathscr{V}$ such that $\mathscr{V}_0$ is finite dimensional and $\mathscr{V}_1 = \mathscr{V}$ and the associated family of functionals

$$J_c := J_{|\mathscr{V}_c}, \ c \in [0, 1]$$

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$$J_c := J_{|\mathscr{V}_c}, \;\; c \in [0, 1]$$

and we recall that

Definition

For $c \in [0, 1]$, a couple $(\alpha,\, \upsilon) \in \mathscr{V}_c$ is a critical point for J_c if

$$DJ_c\left((\alpha,\, \upsilon), \cdot\right) \equiv 0, \;\text{ on }\; \mathscr{V}_c\,.$$

Hestenes

We can now formulate two theorems which can be deduced from the seminal paper by M. Hestenes, Pacific Journal 1951

Theorem

The quadratic form J is positive definite if and only if J_0 is positive definite and for every $c \in (0, 1]$ there is no critical point for J_c.

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Theorem

The quadratic form J is positive semi-definite if and only if J_0 is positive semi-definite and for every $c \in [0, 1)$ there is no critical point for J_c which is not a critical point also for J.

M. Hestenes's Notes

Since y satisfies equations ~~(12.2)~~ (12.8), we have $J(y,v) = 0$. Hence,

$$J(x) = \int_a^b r\, y^2\, \dot{u}^2\, dt \geq 0,$$

the equality holding only in case $\dot{u} = 0$ on $a \leq t \leq b$. Since $u(a) = 0$, we have $x = 0$. This proves the theorem.

THEOREM 12.4. Suppose that $r(t) > 0$ on $a \leq t \leq b$ and let

$$y: \quad y(t) \quad (a \leq t \leq b)$$

be a nonnull solution of the Euler equations (12.8) having $y(a) = 0$ and $y(t) \neq 0$ on $a < t < b$. If $y(b) \neq 0$, then $J(x) > 0$ for all $x \neq 0$ in $\mathcal{B}$. If ~~$y(a)$~~ $y(b)=0$ $= 0$, then $J(x) \geq 0$ for all x in $\mathcal{B}$, the equality holding if and only if x is a scalar multiple of y.

We can suppose that $y(t) > 0$ $a < t < b$. Suppose that $y(b) \neq 0$ and let z be a second solution of (12.8) having $z(a) = 1$. By continuity there is a positive number α such that

$$y(t) + \alpha z(t) > 0 \qquad a \leq t \leq b.$$

A family of subspaces

For $c \in [0, 1]$ fix $t_0 \in [a, b]$ and consider a nested increasing family of closed intervals such that that

1. $I_0 = \{t_0\}, \quad I_1 = [a, b]$

A family of subspaces

We can now define a family of subspaces $\mathscr{V}_c \subseteq \mathscr{V}$, $c \in [0,1]$, as

$$\mathscr{V}_c = \{(\alpha, \upsilon) \in \mathscr{V} : \upsilon(t) \equiv 0 \quad \text{for} \quad t \in [a,\ b] \setminus I_c\}\,.$$

If we denote by $\Sigma_c := \Sigma_{|\mathscr{V}_c}$ then

$$\mathscr{V}_c = \ker \Sigma_c$$

we have that $\mathscr{V}_1 = \mathscr{V}$ and

$$\mathscr{V}_0 = \{(\alpha, 0) \in \mathscr{V} \ \mid N_a\,\alpha = N_b\,\alpha\}$$

A family of subspaces

The subspace $\operatorname{Im}\Sigma_c$ plays the role of the reachable set for the control system therefore we can give the following

Definition

For $c \in [0, 1]$ we say that the control system is c-controllable if

$$\operatorname{Im}\Sigma_c = \mathbb{R}^n.$$

This definition reduces to the known ones in the classical cases.

A family of subspaces

We can also characterize the vectors $\sigma \in \operatorname{Im} \Sigma_c^\perp$

Lemma

$\sigma \in \operatorname{Im} \Sigma_c^\perp$ *if and only if*

$$\sigma B(t) = 0,\ \ t \in [a_c,\, b_c], \quad \sigma(N_a - N_b) = 0.$$

The Hamiltonian

The minimized Hamiltonian associated with our L-Q optimization problem is

$$H_t : \mathbb{R} \times (\mathbb{R}^n)^* \times \mathbb{R}^n \to \mathbb{R}$$

is given by

$$H_t(p, q) := -\frac{1}{2} R^{-1}(t) \left[\left(p\, B(t) + Q(t) q \right) \right]^2$$

The Hamiltonian

The associated Hamiltonian system in $(\mathbb{R}^n)^* \times \mathbb{R}^n$ is

The Jacobi system

$$\begin{pmatrix} \dot{\lambda}(t) \\ \dot{\zeta}(t) \end{pmatrix} = \overrightarrow{H}_t(\lambda(t), \zeta(t))$$

The Hamiltonian

The *optimal control* obtained from (λ, ζ) as

The Feedback

$$\upsilon(t) := -R^{-1}(t)\big(\lambda(t)\,B(t) + Q(t)\zeta(t)\big)$$

will be referred to as the **feedback control**.

c-Transversal Extremal

c-Transversal Extremal

For $c \in [0, 1]$ we will call **c-transversal extremal** a solution of the Jacobi system satisfying

$$\begin{pmatrix} \zeta(a_c) \\ \zeta(b_c) \end{pmatrix} \in \operatorname{Im} \begin{pmatrix} N_a \\ N_b \end{pmatrix}$$

$$\Gamma N^\sharp \begin{pmatrix} \zeta(a_c) \\ \zeta(b_c) \end{pmatrix} = -\lambda(a_c) N_a + \lambda(b_c) N_b,$$

c-Transversal Extremal

For $c \in [0,1]$ if $\sigma \in \operatorname{Im} \Sigma_c^{\perp}$ then $(\sigma, 0)$ is a c-transversal extremal. These extremals are the only ones with a zero state component

Lemma

If (λ, ζ) *is a c-transversal extremal such that* $\zeta(t) = 0, t \in [a_c, b_c]$ *then* $(\lambda, \zeta) = (\sigma, 0)$ *with* $\sigma \in \operatorname{Im} \Sigma_c^{\perp}$.

c-Transversal Extremal

This last Lemma motivates introducing the following

An equivalence relation

For $c \in [0,1]$ we say that two c-transversal extremals (λ_1, ζ_1) and (λ_2, ζ_2) are equivalent if and only if

$$(\lambda_1, \zeta_1) - (\lambda_2, \zeta_2) = (\sigma, 0), \ \sigma \in \operatorname{Im} \Sigma_c^{\perp}$$

c-Transversal Class

This equivalence relation allows us to define

Definition

For $c \in [0, 1]$ we will call **c**-transversal class an equivalence class Θ_c of c- transversal extremals.
The c-transversal class will be said non trivial if $\zeta \neq 0$ on I_c.
The only trivial c-transversal class in the one containing the trivial couple $(0, 0) \in \mathscr{V}$.

A c- transversal class is a linear manifold of dimension $\dim \operatorname{Im} \Sigma_c^\perp$ and then it reduces to a singleton if and only if Σ_c in onto

c-Transversal Class

The next Lemma states that each element of the same class identifies the same admissible $(\alpha, \upsilon) \in \mathscr{V}$.

Lemma

For a given $c \in [0, 1]$ *every* $(\lambda, \zeta) \in \Theta_c$ *identifies the same* $(\alpha, \upsilon) \in \mathscr{V}_c$ *through*

$$\alpha = N^{\sharp} \begin{pmatrix} \zeta(a_c) \\ \zeta(b_c) \end{pmatrix}$$

$$\upsilon(t) = \begin{cases} -R^{-1}(t)\big(\lambda(t)B(t) + Q(t)\zeta(t)\big) & t \in I_c \\ 0 & t \in I \setminus I_c \end{cases}$$

hence $\upsilon(t)$ *is the feedback control on* I_c *and zero outside*

c-Transversal Class

Also the converse is true

Lemma

For $c \in [0,1]$ if the are two c-transversal extremals (λ_1, ζ_1) and (λ_2, ζ_2) which identify the same $(\alpha, \upsilon) \in \mathscr{V}_c$ then they belong to the same class Θ_c.

c-Transversal Class

The values of the control are identified by the c-transversal extremal through the feedback only on I_c but it may happen that the same holds true on the whole interval I.

Definition

A c-transversal class Θ_c is **singular** if for every $(\lambda, \zeta) \in \Theta_c$ in the class we have that

$$\upsilon(t) = -R^{-1}(t)\big(\lambda(t)B(t) + Q(t)\zeta(t)\big) = 0, \ \ t \in [a, b] \setminus I_c$$

c-Transversal Class and Critical Points

A c-transversal class Θ_c represents an $(\alpha, \upsilon) \in \mathscr{V}_c$, but only some $(\alpha, \upsilon) \in \mathscr{V}_c$, are represented by a c-transversal class

Theorem

For $c \in [0, 1]$, $(\alpha, \upsilon) \in \mathscr{V}_c$ is a critical point for J_c if and only if it is represented by c-transversal class Θ_c, moreover $(\alpha, \upsilon) \in \mathscr{V}_c$ is non-zero if and only if c-transversal class Θ_c in non trivial

c-Transversal Class

We can reformulate Theorem 4 and 5 in Hamiltonian terms

Theorem

The quadratic form J is positive definite if and only if J_0 is positive definite and there exist no non-trivial c-transversal class.

c-Transversal Class

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Theorem

The quadratic form J is positive semi-definite if and only if J_0 is positive semi-definite and every non-trivial c-transversal class is singular.

Separate Boundary Conditions

Let us examine the classical case of separate boundary condition on the two end-points of the trajectory that is the case when we can write $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$ with $k_1 + k_2 = k$ and

$$N_a \, \alpha = ({}^{\mathrm{s}}N_a \; 0) \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = {}^{\mathrm{s}}N_a \, \alpha_1, \quad N_b \, \alpha = (0 \;\; {}^{\mathrm{s}}N_b) \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = {}^{\mathrm{s}}N_b \, \alpha_2,$$

and the end-point cost is

$$\Gamma = \begin{pmatrix} \Gamma_a & 0 \\ 0 & \Gamma_b \end{pmatrix}$$

$$\Gamma[\alpha]^2 = \Gamma_a[\alpha_1]^2 + \Gamma_b[\alpha_2]^2$$

Separate Boundary Conditions

Necessary conditions
Let

$$W_0 = W_{0a} \oplus W_{0b} := \ker N_a \oplus \ker N_b$$

clearly

$$\mathscr{W}_0 = \{(\alpha, \beta, 0) \mid (\alpha, \beta) \in W_0\} \subseteq \mathscr{V}$$

Assumption

$$W_{0a} = \{0\} \quad \textit{and} \quad W_{0b} = \{0\}$$

Separate Boundary Conditions

The above assumption as an interesting consequence

- N_a admits a left inverse $N_a^\sharp$ such that $N_a^\sharp N_b = Id_{\mathbb{R}^h}$.
- N_b admits a left inverse $N_b^\sharp$ such that $N_b^\sharp N_b = Id_{\mathbb{R}^k}$.

Separate Boundary Conditions

The boundary conditions on I_c become

$$\zeta(a_c) \in \operatorname{Im} {}^{\mathrm{s}}N_a, \quad \zeta(b_c) \in \operatorname{Im} {}^{\mathrm{s}}N_b$$
$$\Gamma_a \, {}^{\mathrm{s}}N_a^{\sharp}\zeta(a_c) = -\lambda(a_c) \, {}^{\mathrm{s}}N_a \quad \Gamma_b \, {}^{\mathrm{s}}N_b^{\sharp}\zeta(b_c) = \lambda(b_c) \, {}^{\mathrm{s}}N_b$$

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$$\Gamma_a \, {}^{\mathrm{s}}N_a^\sharp \zeta(a_c) = -\lambda(a_c) \, {}^{\mathrm{s}}N_a \quad \Gamma_b \, {}^{\mathrm{s}}N_b^\sharp \zeta(b_c) = \lambda(b_c) \, {}^{\mathrm{s}}N_b$$

Initial and Final Subspaces

$$L_a = \left\{ \left(-\Gamma_a({}^{\mathrm{s}}N_a^\sharp x, {}^{\mathrm{s}}N_a^\sharp(\cdot)) + \sigma_a, x \right) | x \in \operatorname{Im} {}^{\mathrm{s}}N_a, \sigma_a \in (\operatorname{Im} \; {}^{\mathrm{s}}N_a)^\perp \right\}$$

$$L_b = \left\{ \left(\Gamma_b({}^{\mathrm{s}}N_b^\sharp y, {}^{\mathrm{s}}N_b^\sharp(\cdot)) + \sigma_b, y \right) | y \in \operatorname{Im} {}^{\mathrm{s}}N_b \, \sigma_b \in (\operatorname{Im} \; {}^{\mathrm{s}}N_b)^\perp \right\}$$

which are Lagrangian subspaces

c-Transversal Extremal

c- transversal extremals

For $c \in [0, 1]$ we will call **c-transversal extremal** a solution of the Jacobi system satisfying

$$(\lambda(a_c), \zeta(a_c)) \in L_a, \ (\lambda(b_c), \zeta(b_c)) \in L_b$$

or, equivalently

$$\mathscr{H}_{(a_c,b_c)}(\lambda(a_c), \zeta(a_c)) \in L_b,$$

or, equivalently

$$\mathscr{H}_{(b_c,a_c)}(\lambda(b_c), \zeta(b_c)) \in L_a$$

The c-transversal extremal will be said **non trivial** if $\zeta \neq 0$ on I_c.

The Riccati Equation

Let us consider the Riccati differential equation

The Riccati Equation

$$\dot{W}(t) = \Big(Q^*(t) + W(t)B(t)\Big)R^{-1}(t)\Big(B^*(t)W(t) + Q(t)\Big)$$

Where W is a time dependent quadratic form

$$W : R^n \times I \subset \mathbb{R} \to \mathrm{R^n}^*$$

Thanks to its special structure if W is a solution then also W^* is a solution. Where W^* is defined by

$$\langle W^* x,\, y\rangle = \langle Wy,\, x\rangle$$

The Riccati Equation - Fixed-Free

Let us now consider the following boundary conditions

$$\xi(a) = 0, \quad \xi(b) = {}^{\mathrm{s}}N_b \alpha_2$$

where we assume that $\operatorname{Im} {}^{\mathrm{s}}N_b = \mathbb{R}^n$ hence from our assumptions it follows that $N_a \equiv 0 : \{0\} \to \mathbb{R}^n$ and N_b is invertible.

The Riccati Equation - Fixed-Free

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Initial and Final Lagrangian Subspaces

$$L_a = \{(\sigma_a, 0) \mid \sigma_a \in \mathbb{R}^{n*}\}$$

$$L_b = \left\{ \left(\Gamma_b\big({}^{\mathrm{s}}N_b^{-1} x,\ {}^{\mathrm{s}}N_b^{-1}(\cdot)\big),\ x \right) \mid x \in \mathbb{R}^n \right\}$$

The Riccati Equation

Theorem

The form J is positive definite if and only if there exists a symmetric solution of the Riccati equation on $[a, b]$

$$W : [a, b] \to \mathscr{L}(\mathbb{R}^n, \mathrm{R^n}^*)$$

such that

$$W(b) = \Gamma_b\left({}^{\mathrm{s}}N_b^{-1}(\cdot),\ {}^{\mathrm{s}}N_b^{-1}(\cdot)\right)$$

The Riccati Equation - Subspace-Free

Let us now consider the following boundary conditions

$$\xi(a) = {}^{\mathrm{s}}N_a\alpha_1, \quad \xi(b) = {}^{\mathrm{s}}N_b\alpha_2$$

where we assume that $\operatorname{Im} {}^{\mathrm{s}}N_b = \mathbb{R}^n$ hence from our assumptions it follows that N_a is injective and N_b is invertible.

The Riccati Equation - Subspace-Free

Let us now consider the following boundary conditions

$$\xi(a) = {}^{\mathrm{s}}N_a\alpha_1, \quad \xi(b) = {}^{\mathrm{s}}N_b\alpha_2$$

where we assume that $\operatorname{Im} {}^{\mathrm{s}}N_b = \mathbb{R}^n$ hence from our assumptions it follows that N_a is injective and N_b is invertible.

Initial and Final Lagrangian Manifolds

$$L_a = \left\{ \left(-\Gamma_a({}^{\mathrm{s}}N_a^{\sharp}x, {}^{\mathrm{s}}N_a^{\sharp}(\cdot)) + \sigma_a, x \right) | x \in \operatorname{Im} {}^{\mathrm{s}}N_a, \sigma_a \in (\operatorname{Im} {}^{\mathrm{s}}N_a)^{\perp} \right\}$$

$$L_b = \left\{ \left(\Gamma_b\left({}^{\mathrm{s}}N_b^{-1}x, {}^{\mathrm{s}}N_b^{-1}(\cdot)\right), x \right) \mid x \in \mathbb{R}^n \right\}$$

Riccati

To reduce this problem to the previous we consider the sub-problem obtained by taking $\xi(a) = 0$ or, in other words let us consider the restriction of J to the subspace

$$\mathscr{V}_1 = \{(0, \alpha_2, v) \in \mathscr{V}\}$$

The Riccati Equation

Lemma

$(\alpha_1, \alpha_2, \upsilon) \in \mathscr{V}_1^{\perp_J}$ if and only if there is a solution (λ, ζ) of the Jacobi system with boundary conditions

$$\zeta(a) = {}^{s}N_a\alpha_1,\ \zeta(b) = N_b\alpha_2$$
$$\eta(b) = \Gamma_b\Big(\alpha_2, N_b^{-1}(\cdot)\Big)$$

and such that

$$\upsilon(t) = -R^{-1}(t)\big(Q(t)\zeta(t) + \lambda(t)B(t)\big)$$

Riccati

Theorem

The form J is positive definite if and only if there exists a symmetric solution of the Riccati equation on $[a, b]$

$$W : [a, b] \to \mathscr{L}(\mathbb{R}^n, \mathrm{R^n}^*)$$

such that

$$W(b) = \Gamma_b\left({}^{\mathrm{s}}N_b^{-1}(\cdot),\ {}^{\mathrm{s}}N_b^{-1}(\cdot)\right) \tag{2}$$

$${}^{\mathrm{s}}N_a^* W(a)\, {}^{\mathrm{s}}N_a + \Gamma_a > 0 \tag{3}$$

Riccati

To tackle the final general case we use the following Theorem by M. Hestenes

Theorem

Let J be a Legendre form and let $K(x)$ be a w-continuous form with the property that $J(x) > 0$ whenever $K(x) = 0$ and $x \neq 0$. Then there is a number β such that $J(x) + \frac{1}{2}\beta K(x)$ is positive definite. If $K(x) \geq 0$ whenever $J(x) \leq 0$, the number β can be chosen to be positive.

Riccati

Hence we can add a penalty term to have a problem with a free final point.
Let $\omega_1, \ldots, \omega_r$ bas for $\operatorname{Im} {}^{\mathrm{s}}N_b^{\perp}$, and define

$$\Lambda[\xi(b)]^2 := \frac{1}{2}\beta \sum_{1=1}^{r} \omega_i \otimes \omega_i[\xi(b)]^2$$

Back to the initial example $N_0 := \{x_0\}, N_T := M$

The extended second variation

$$J_{\mathrm{ext}}[(\varepsilon_0, \delta x, \varepsilon_1, \upsilon)]^2 =$$

$$= \frac{\varepsilon_0^2}{2}\left(L_k^2 c\left(\widehat{x}_b\right) + H_{12}(\widehat{\ell}_1)\right) - \varepsilon_1 L_{\delta x} L_{f_1} c\left(\widehat{x}_b\right) - \frac{\varepsilon_1^2}{2} L_{f_1}^2 c\left(\widehat{x}_b\right) +$$

$$+ \frac{1}{2}\int_a^b \left\{2\upsilon(t) L_{\zeta(t)} L_{\dot{g}_t} c\left(\widehat{x}_b\right) + \upsilon(t)^2 R(t)\right\} \mathrm{d}t$$

Back to the initial example $N_0 := \{x_0\}, N_T := M$

The extended second variation

$$J_{\text{ext}}[(\varepsilon_0, \delta x, \varepsilon_1, \upsilon)]^2 =$$
$$= \frac{\varepsilon_0^2}{2}\left(L_k^2 c\,(\widehat{x}_b) + H_{12}(\widehat{\ell}_1)\right) - \varepsilon_1 L_{\delta x} L_{f_1} c\,(\widehat{x}_b) - \frac{\varepsilon_1^2}{2} L_{f_1}^2 c\,(\widehat{x}_b) +$$
$$+ \frac{1}{2}\int_a^b \left\{2\upsilon(t) L_{\zeta(t)} L_{\dot{g}_t} c\,(\widehat{x}_b) + \upsilon(t)^2 R(t)\right\} \mathrm{d}t$$

subject to

$$\dot{\zeta}(t) = \upsilon(t)\dot{g}_t(\widehat{x}_b), \qquad \zeta(a) = \varepsilon_0\, k(\widehat{x}_b), \qquad \zeta(b) = \delta x + \varepsilon_1 f_1(\widehat{x}_b)$$

Back to the initial example $N_0 := \{x_0\}, N_T := M$

The *admissible variations* are given by $(\varepsilon_0, \delta x, \varepsilon_1, \upsilon) \in T_{\widehat{x}_b} M \times \mathbb{R} \times \mathbb{R} \times L^2([a,b])$ such that

$$\int_a^b \{\upsilon(t)\dot{g}_t(\widehat{x}_b)\}\,\mathrm{d}t - \delta x - \varepsilon_1 f_1(\widehat{x}_b) + \varepsilon_0\, k(\widehat{x}_b) = 0$$

Assumption

Assume

$$H_{12}(\widehat{\ell}_1) > 0$$

Positivity of J_{ext}

Left end-point. $\alpha_1 := \varepsilon_0$

$$^{\mathrm{s}}N_a \alpha_1 = \varepsilon_0\, k(\widehat{x}_b), \quad \mathbb{R}^{k_1} = \mathbb{R}$$

Decouplig

$$\ker {}^{\mathrm{s}}N_a = W_{0a} = \{\varepsilon_0 \in \mathbb{R} : \varepsilon_0\, k(\widehat{x}_b) = 0\} = \begin{cases} \mathbb{R} & \text{if } k(\widehat{x}_b) = 0 \\ \{0\} & \text{if } k(\widehat{x}_b) \neq 0 \end{cases}$$

Positivity of J_{ext}

$$\Gamma_a \alpha_1 = \varepsilon_0^2 \left(L_k^2 c\left(\widehat{x}_b\right) + H_{12}(\widehat{\ell_1}) \right)$$

Since $H_{12}(\widehat{\ell_1}) > 0$ then

Decoupling

$$Z_{0a} = \begin{cases} \{0\} & \text{if } k(\widehat{x}_b) = 0 \\ \{0\} & \text{if } k(\widehat{x}_b) \neq 0 \end{cases}$$

$$Z_{1a} = \begin{cases} \mathbb{R} & \text{if } k(\widehat{x}_b) = 0 \\ \{0\} & \text{if } k(\widehat{x}_b) \neq 0 \end{cases}$$

Positivity of J_{ext}

Necessary conditions for positive definiteness

$$\Gamma_{a|W_{0a}} > 0 \quad \text{always}$$

Positivity of J_{ext}

Right end-point. $\alpha_2 = (\delta x, \varepsilon_1)$

$$N_b \alpha_2 = \delta x + \varepsilon_1 f_1(\widehat{x}_b), \quad \mathbb{R}^{k_2} = T_{\widehat{x}_b} M \times \mathbb{R}$$

$$\ker {}^{\mathrm{s}}N_b = W_{0b} = \{(\delta x,\, \varepsilon_1) \in T_{\widehat{x}_b} M \times \mathbb{R} : \delta x + \varepsilon_1 f_1(\widehat{x}_b) = 0\} =$$
$$= \begin{cases} \{0\} \times \mathbb{R} & \text{if } f_1(\widehat{x}_b) = 0 \\ span\{(-f_1(\widehat{x}_b),\, 1)\} & \text{if } f_1(\widehat{x}_b) \neq 0 \end{cases}$$

Positivity of J_{ext}

$$\Gamma_b\, \alpha_2 = -\varepsilon_1 L_{\delta x} L_{f_1} c\,(\widehat{x}_b) - \frac{\varepsilon_1^2}{2} L_{f_1}^2 c\,(\widehat{x}_b)$$

- when $f_1(\widehat{x}_b) = 0$ we have $\Gamma_b\, \alpha_2 = 0$
- when $f_1(\widehat{x}_b) \neq 0$ we have

$$\Gamma_b\, [(-f_1(\widehat{x}_b),\, 1)]^2 = L_{f_1}^2 c\,(\widehat{x}_b)$$

and hence we must assume $\boxed{L_{f_1}^2 c\,(\widehat{x}_b) > 0}$

Positivity of J_{ext}

$$Z_{0b} = \begin{cases} \{0\} \times \mathbb{R} & \text{if } f_1(\widehat{x}_b) = 0 \\ \{0\} & \text{if } f_1(\widehat{x}_b) \neq 0 \end{cases}$$

$$Z_{1b} = \begin{cases} \{0\} & \text{if } f_1(\widehat{x}_b) = 0 \\ span\{(-f_1(\widehat{x}_b), 1)\} & \text{if } f_1(\widehat{x}_b) \neq 0 \end{cases}$$

Positivity of J_{ext}

Necessary conditions for positive definiteness

- when $f_1(\widehat{x}_b) = 0$ they are NOT satisfied
- when $f_1(\widehat{x}_b) \neq 0$ we have that $\Gamma_{b|W_{0b}} > 0$ assuming $L^2_{f_1} c\,(\widehat{x}_b) > 0$

Positivity of J_{ext} - Decoupling

• If $k(\widehat{x}_b) = 0$ then $Z_{0a} = \{0\}$ and $Z_{1a} = W_{0a} = \mathbb{R}$ and $W_{1a} = \{0\}$
In this case the boundary condition $\zeta(a) = \varepsilon_0 \, k(\widehat{x}_b)$, does not depend on ε_0 and we can decouple the quadratic part in ε_0

Positivity of $J_{\rm ext} = J^1_{\rm ext} + J^2_{\rm ext}$ - Decoupling

Decoupling

$$J^1_{\rm ext}[\varepsilon_0]^2 = \frac{\varepsilon_0^2}{2} H_{12}(\widehat{\ell}_1) > 0$$

$$J^2_{\rm ext}[(\delta x, \varepsilon_1, w)]^2 = -\varepsilon_1 L_{\delta x} L_{f_1} c\,(\widehat{x}_b) - \frac{\varepsilon_1^2}{2} L^2_{f_1} c\,(\widehat{x}_b) + \\ + \frac{1}{2}\int_a^b \left\{ 2\upsilon(t) L_{\zeta(t)} L_{\dot{g}_t} c\,(\widehat{x}_b) + \upsilon(t)^2 R(t) \right\} \mathrm{d}t$$

subject to

$$\dot{\zeta}(t) = \upsilon(t)\dot{g}_t(\widehat{x}_b), \quad \zeta(a) = 0, \quad \zeta(b) = \delta x + \varepsilon_1 f_1(\widehat{x}_b)$$

Positivity of J_{ext} - Decoupling

• If $k(\widehat{x}_b) \neq 0$ then $Z_{0a} = Z_{1a} = W_{0a} = \{0\}$ and decoupling is not needed.

Positivity of J_{ext} - Decoupling

- If $f_1(\widehat{x}_b) = 0$ then $Z_{0a} = \{0\} \times \mathbb{R}$ and J_{ext} cannot be positive

Positivity of J_{ext} - Decoupling

• If $f_1(\widehat{x}_b) \neq 0$ we assume $L^2_{f_1} c\,(\widehat{x}_b) > 0$ then
$Z_{0b} = \{0\}$ and $Z_{1b} = W_{0b} = span\{(-f_1(\widehat{x}_b),\, 1)\}$

$$W_{1b} = \{(\delta x,\, \varepsilon_1)\colon\, L_{\delta x} L_{f_1} c\,(\widehat{x}_b) = 0,\, \varepsilon_1 \in \mathbb{R}\}$$

Positivity of $J_{\rm ext}$ - Decoupling

Decoupling

$$J^1_{\rm ext}[\varepsilon_1]^2 = \frac{\varepsilon_1^2}{2} L^2_{f_1} c\left(\widehat{x}_b\right) > 0$$

$$J^2_{\rm ext}[(\varepsilon_0, \delta x, \upsilon)]^2 = \frac{\varepsilon_0^2}{2}\left(L^2_k c\left(\widehat{x}_b\right) + H_{12}(\widehat{\ell}_1)\right) - \frac{\left(L_{\delta x} L_{f_1} c\left(\widehat{x}_b\right)\right)^2}{2 L^2_{f_1} c\left(\widehat{x}_b\right)}$$

$$+ \frac{1}{2}\int_a^b \left\{2\,\upsilon(t) L_{\zeta(t)} L_{\dot{g}_t} c\left(\widehat{x}_b\right) + \upsilon(t)^2 R(t)\right\} \mathrm{d}t$$

subject to

$$\dot{\zeta}(t) = \upsilon(t)\dot{g}_t(\widehat{x}_b), \qquad \zeta(a) = \varepsilon_0\, k(\widehat{x}_b), \qquad \zeta(b) = \delta x$$

THANKS FOR YOUR ATTENTION