Mathematical Control Theory
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On the Pontryagin maximum principle

A "classical" Pontryagin maximum principle (PMP)

V. G. Boltyanskii (1957), L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, E.F Mischenko (1961)

- finite-dimensional state space
- needle variations
- set approximation by Boltyanskii cones
- separation theorem

H. Halkin (1964), A. Ya. Dubovitskii, A. A. Milyutin (1965), J. Warga (1972), L. W. Neustadt (1976), A. D. Ioffe, V. M. Tihomirov (1979)

The key idea is to show that if a convex approximation to some "target set" intersects the interior of a convex approximation to the reachable set at some point, then the reachable set has a nontrivial intersection with the target set.

A "very smooth" Pontryagin maximum principle (PMP)

A. Krener (1977), A. Agrachev, R. Gamkrelidze (1979),
H. W. Knobloch (1981), H. Schättler (1986), A. Bressan (1985),
R. M. Bianchini, G. Stefani (1993), M. Kawski, H. Sussmann (1997),
A. Agrachev, G. Stefani, P. L. Zezza (1999)

- finite-dimensional state space
- smooth control systems
- exponental representation of flows
- second- and higher order optimality conditions
- the Legendre-Clebsh, Kelley and Goh optimality conditions

“Non-smooth versions” of the PMP

J. Warga (1976 and 1983)

- Lipschitz continuous data
- “a derivate container” as a generalized derivative

“Lojasiewicz refinement”: the data are assumed to be differentiable only at the optimal state trajectory

- H. J. Sussmann (1999) proved a non-smooth version based on semidifferentials for extensions to nondifferentiable data, separation and transversality theorems giving necessary conditions for two sets to be separated, and an extension theorem from homotopy theory to set-valued maps about the existence of connected sets of zeros.
- A. D. Ioffe (2011) proposed a “non-variational” approach for obtaining necessary optimality conditions based on an abstract result called the “optimality alternative” for locally Lipschitz functions.

Infinite-dimensional versions of the PMP

Contributors

A. G. Butkovsky (1961) $\Rightarrow$ for systems governed by integral equations

G. L. Kharatishvili (1961) $\Rightarrow$ for systems by ordinary delay equations

Yu. V. Egorov (1963) $\Rightarrow$ constructed an example showing that the maximum principle does not generally hold for infinite-dimensional systems

Important!

Necessary optimality conditions of the form of Pontryagin maximum principle can be proved

only under additional assumptions !!!

Counter example of Egorov (1963)

Let the phase space be the space l_2 consisting of all sequences $x = (x_1, ..., x_n, ...)$ with the norm

$$||x|| = \left(\sum_{n=1}^{\infty} x_n^2 \right)^{1/2},$$

and let the set of admissible controls be the set of all piece-wise continuous functions $u : [0, T] \to U$, where $T > 0$ and

$$U = \left\{ (u_1, ..., u_n, ...) \in l_2 : |u_n| \leq \frac{1}{n} + \frac{1}{n^2} \right\}.$$

Counter example of Egorov (1963)

Consider the following time-optimal problem

$$T \rightarrow \min$$

subject to the differential equation

$$\frac{dx(t)}{dt} = u(t)$$

with the initial and final conditions

$$x(0) = x_0 := (0, ..., 0, ...), \quad x(T) = x_1 := (1, 1/2, ..., 1/n, ...).$$

Counter example of Egorov (1963)

We first show that the control $\bar{u}(t) \equiv (1, 1/2, ..., 1/n, ...)$ is optimal. Indeed, for this control we have

$$x(t) = (t, t/2, ..., t/n, ..)$$

and for $T := 1$ we have that $x(T) = x_1$.

On the other hand the minimal time to pass from $x_n(0) = 0$ to $x_n(T) = 1/n$ is equal to $\dfrac{n}{n+1}$, achieved by $u_n(t) \equiv \dfrac{1}{n} + \dfrac{1}{n^2}$. As n increases infinitely, we see that we can not pass from x_0 to x_1 for a time, which is strictly less than 1.

If the maximum principle were valid, then there would exist a non-zero constant vector $\psi = (\psi_0, \psi_1, ..., \psi_n, ...) \in l_2$ such that the function

$$H(\psi(t), x(t), u) = \psi_0 + \sum_{n=1}^{\infty} \psi_n u_n$$

would attain its maximum with respect to $u \in U$ at $\bar{u}_n(t) = \frac{1}{n}$. However, if $\psi_k \neq 0$ for some $k \geq 1$, then the function $H(\psi(t), x(t), u)$ has its maximal value in the class of all $u \in U$ for

$$u_k = \left(\frac{1}{k} + \frac{1}{k^2}\right) \textit{sign}\ \psi_k$$

only. So, $\bar{u}_n(t) = \frac{1}{n}$ for all $n \geq 1$ implies $\psi_n = 0$ for all $n \geq 1$.
Since the end time T is not fixed, the maximized Hamiltonian must be identically zero, and hence $\psi_0 = 0$.

Infinite-dimensional version of the Pontryagin maximum principle (after the Egorovs example)

Contributors

A.I. Egorov (1967) $\Rightarrow$ for systems governed parabolic and hyperbolic systems with terminal state constrained by finitely many equalities

X. Li and Y. Yao (1985), H. Fattorini (1987), H. Fattorini and H. Frankowska (1991), X. Li, Y. Yao, J. Yong (1994), E. Casas, J.-P. Raymond, H. Zidani (2000), N. Arada, J.-P. Raymond (2002), B. Mordukhovich, J.-P. Raymond (2005) $\Rightarrow$ for optimal control problems in Banach spaces, assuming finite codimensionality in the state space of the target set or of some other set closely related to it

M.K., N. Ribarska, Ts. Tsachev (2011) $\Rightarrow$ drop the key assumption that the set of variations (in the state space) of the state trajectory's endpoint (resulting from the control variations) is finite codimensional.

Infinite-dimensional version of the Pontryagin maximum principle (after the Egorovs example)

Age-structured systems

M. Brokate (1985), G. Feichtinger, G. Tragler, V. Veliov (2003), S. Faggian (2008), N. Osmolovskii, V. Veliov (2017)
- needle variations and separation theorem
- an abstract Lagrange multiplier theorem (for the case of the presence of mixed control-state constraints)

“Non-smooth versions” of the PMP

F. Clarke (1976)

Main features:
- simple variational arguments imply necessary optimality conditions for control problems with no terminal constraints;
- finding a neighboring process that is a minimizer for a perturbed problem with free right endpoint;
- necessary optimality conditions for the original problem are obtained by passing to the limit.

"Non-smooth versions" of the PMP

In a series of papers (cf., e.g., H. J. Sussmann (2006)) the corresponding proofs are based on separation theorems for sets.

In the nonsmooth versions of the Pontryagin Maximum Principle, the used normal cone used is the polar of the Boltyanskii approximating cone. All these versions can fail to be true if the Clarke normal cone (and, any smaller normal cone, such as the Mordukhovich cone) is used instead. The key fact is A. Bressan's recent example of
a) two closed subsets A_1 and A_2 of R^n that intersect only at a point p;
b) A_1 has a Boltyanskii approximating cone C_1 at p;
c) A_2 has a Clarke tangent cone C_2 at p;
d) the cones C_1 and C_2 are strongly transversal, i.e. $C_1 - C_2 = R^n$ and $C_1 \cap C_2 \neq \emptyset$.

An abstract Lagrange multiplier rule

In a series of papers (cf., e.g., H. J. Sussmann (2006)) the corresponding proofs are based on nonseparation theorems for sets.

Definition

Let X be a Banach space and A, B be closed subsets of X. It is said that A, B are locally separated at x_0 if there exists a neighborhood Ω of x_0 such that $\Omega \cap A \cap B = \{x_0\}$.

Let us approximate A (resp. B) by some "tangent cone" C^A (resp. C^B).

Definition

The cones C^A and C^B are said to be strongly transversal iff $C^A - C^B = X$ and $C^A \cap C^B \neq \{0\}$.

An abstract Lagrange multiplier rule

A nonseparation theorem should assert that a necessary condition for the sets A and B to be locally separated at x_0 is that C^A and C^B are not strongly transversal.

Finite dimensional case

The corresponding nonseparation theorems for a finite dimensional space X are true if both "tangent cones" are interpreted to mean "Boltyanskii approximating cones" and also if they both are taken to mean "Clarke tangent cones". A. Bressan has constructed an example of two four-dimensional separated sets A and B at x_0, where (surprisingly) the approximating cones C^A and C^B are strongly transversal (C^A and C^B approximate the sets A and B in the Boltyanskii, respectively, in the Clarke sense at x_0).

In infinite dimensional setting, things become even worse: there exist convex sets A and B that are locally separated at a common point x_0 such that the corresponding approximating Clarke tangent cones C^A and C^B of the sets A and B, respectively, at x_0 are strongly transversal. (Take a Hilbert cube and a ray.)

This motivates the search for a smaller approximating cone than the Clarke one such that the strong transversality of two approximating cones at a common point of two sets implies local nonseparation of these sets. We proposed an approach whose main idea is that the uniformity of a tangent cone is crucial for obtaining necessary optimality conditions in infinite-dimensional setting.

An abstract Lagrange multiplier rule

A uniform tangent set

Let S be a closed subset of X and x_0 belongs to S. We say that the bounded set D is a uniform tangent set to S at the point x_0 if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $v \in D$ and for each point $x \in S \cap (x_0 + \delta \bar{\mathbf{B}})$ one can find $\lambda > 0$ for which $S \cap (x + t(v + \varepsilon \bar{\mathbf{B}}))$ is non empty for each $t \in [0, \lambda]$.

A sequence uniform tangent set

Let S be a closed subset of X and x_0 belongs to S. We say that the bounded set D is a sequence uniform tangent set to S at the point x_0 if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $v \in D$ and for each point $x \in S \cap (x_0 + \delta \bar{\mathbf{B}})$ one can find a sequence of positive reals $t_m \longrightarrow 0$ for which $S \cap (x + t_m(v + \varepsilon \bar{\mathbf{B}}))$ is non empty for each positive integer m.

An abstract Lagrange multiplier rule

A (sequence) uniform tangent cone

Let S be a closed subset of X and x_0 belong to S. We say that the cone C is an uniform tangent cone to S at the point x_0 if $C \cap \bar{\mathbf{B}}$ is an uniform tangent set to S at the point x_0. We say that the cone C is a sequence uniform tangent cone to S at the point x_0 if $C \cap \bar{\mathbf{B}}$ is a sequence uniform tangent set to S at the point x_0.

It is remarkable that in finite-dimensional space X the two notions introduced in the above written definition coincide. Moreover, the usual Clarke tangent cone is a (sequence) uniform cone (cf. Rockafellar (1979)).

Some properties: The closure of a (sequence) uniform tangent set D to S at x_0 is an uniform tangent set to S at x_0. The convex hull of a uniform tangent set D is a uniform tangent set to S at x_0.

An abstract Lagrange multiplier rule

Nonseparation theorem

Let A and B be closed subsets of X with $x_0 \in A \cap B$. Let C^A be an uniform tangent cone to the set A at the point x_0 and C^B be a sequence uniform tangent cone to the set B at the point x_0, such that the following conditions hold true:

(A1) There exist $M > 1$ and $\varepsilon \in (0, 1/(9 + 8M))$ such that the set $\left(C^A \cap M\bar{\mathbf{B}}\right) - \left(C^B \cap M\bar{\mathbf{B}}\right)$ is ε-dense in the unit sphere $\{w \in X : \|w\| = 1\}$, that is for every $v \in X$ with $\|v\| = 1$ there exist $v_1 \in C^A \cap M\bar{\mathbf{B}}$ and $v_2 \in C^B \cap M\bar{\mathbf{B}}$ such that $\|v - (v_1 - v_2)\| < \varepsilon$;

(A2) there exists v_0 with unit norm which is ε-close to C^A and to C^B (that is, there are $\tilde{v}_0^A \in C^A$ and $\tilde{v}_0^B \in C^B$ of norm one such that $\|\tilde{v}_0^A - v_0\| < \varepsilon$ and $\|\tilde{v}_0^B - v_0\| < \varepsilon$).

Then for each positive integer n there exists $\bar{x} \neq x_0$ that belongs to the set $A \cap B$ and $\|\bar{x} - x_0\| < 1/n$.

An abstract Lagrange multiplier rule

The above written theorem remains valid if one replaces $C^A \cap \ M\bar{\mathbf{B}}$ (resp. $C^B \cap \ M\bar{\mathbf{B}}$) by a uniform tangent set to A at x_0 contained in $M\bar{\mathbf{B}}$ (resp. by a sequence uniform tangent set to B at x_0 contained in $M\bar{\mathbf{B}}$).

Another property of uniform tangent sets: If A is a closed convex set containing x_0, then the set $(A - x_0) \cap M\bar{\mathbf{B}}$ is a uniform tangent set to A at the point x_0.

It is straightforward to obtain as a corollary the following well-known result (note that it is not easy to find a simple self-contained proof in the literature):

Let X be a finite-dimensional vector space, A and B be closed subsets of X with $x_0 \in A \cap B$. Let C^A and C^B be the Clarke tangent cones to A and B, respectively, at x_0. Let the cones C^A and C^B be strongly transversal. Then the sets A and B are not locally separated at x_0.

An abstract Lagrange multiplier rule

Let us consider the following optimization problem

$$\varphi(G(x)) \to \min \tag{1}$$

subject to:

$$G(x) \in S. \tag{2}$$

Here X is a complete metric space, Y is a Banach space, S is a closed subset of Y, $G : X \to Y$ and $\varphi : Y \to R$ are maps. We consider the Banach space $Y \times R$ equipped with the norm $\|(y, r)\| := \max\{\|y\|, |r|\}$.

Definition of quasisolid sets

Let Y be a Banach space and S be a subset of Y. The set S is said to be quasisolid if its closed convex hull $\overline{\mathrm{co}}\ S$ has nonempty interior in its closed affine hull, i.e. if there exists a point $y_0 \in \overline{\mathrm{co}}\ S$ such that $\overline{\mathrm{co}}\ \{S - y_0\}$ has nonempty interior in $\overline{\mathrm{span}}\ (S - y_0)$ (the closed subspace spanned by $S - y_0$).

An abstract Lagrange multiplier rule

Theorem (Lagrange multiplier rule)

Let $\bar{x}$ be a solution of the problem (1)–(2). We set $\bar{y} := G(\bar{x})$, $\tilde{S} := S \times (-\infty, \varphi(\bar{y})]$ and $\tilde{\mathcal{R}} := \{(G(x), \varphi(G(x)) : \; x \in X\} \subset Y \times R$. Let C^S be an uniform tangent cone to S at the point $\bar{y}$, $C^{\tilde{S}} = C^S \times (-\infty, 0]$ and $C^{\tilde{\mathcal{R}}}$ be a uniform tangent cone to the set $\tilde{\mathcal{R}}$ at the point $\tilde{y} := (\bar{y}, \varphi(\bar{y}))$. We assume that the set $\left(C^{\tilde{S}} \cap \; \tilde{\mathbf{B}}\right) - \left(C^{\tilde{\mathcal{R}}} \cap \; \tilde{\mathbf{B}}\right)$ is quasisolid, where $\tilde{\mathbf{B}}$ is the closed unit ball in $Y \times R$. Then there exists a pair $(\xi, \eta) \in Y^* \times R$ such that

(i) $(\xi, \eta) \neq (\mathbf{0}, 0)$;

(ii) $\eta \in \{0, 1\}$;

(iii) ξ belongs to the polar cone of the cone C^S;

(iv) $-(\xi, \eta)$ belongs to the polar cone of the cone $C^{\tilde{\mathcal{R}}}$.

Remark.

The assertion of the previous Theorem (Lagrange multiplier rule) remains true if one of the approximating cones is a convex sequence uniform tangent cone. The same remark applies also to next assertions.

An abstract Lagrange multiplier rule

Corollary.

Let $\bar{x}$ be a solution of the problem (1)–(2) and let φ be strictly differentiable at $\bar{y}$. Let C^S be an uniform tangent cone to S at the point $\bar{y}$ with $\bar{y} = G(\bar{x})$ and $C^{\mathcal{R}}$ be a uniform tangent cone to the set $\mathcal{R} := \{G(x) : \ x \in X\}$ at the point $\bar{y}$. Setting $C^{\tilde{\mathcal{R}}} := \{(v, \varphi'(\bar{y})v) : \ v \in C^{\mathcal{R}}\}$ and $C^{\tilde{S}} := \{(v, r) : \ v \in C^S, \ r \le 0\}$, we assume that the set

$$\left(C^{\tilde{S}} \cap \ \tilde{\mathbf{B}}\right) - \left(C^{\tilde{\mathcal{R}}} \cap \ \tilde{\mathbf{B}}\right)$$

is quasisolid. Then there exist a nontrivial pair $(\xi, \eta) \in Y^* \times R$ such that

(i) $(\xi, \eta) \neq (\mathbf{0}, 0)$;

(ii) $\eta \in \{0, 1\}$;

(iii) ξ belongs to the polar cone of the cone C^S;

(iv) $-\xi - \eta\varphi'(\bar{y})$ belongs to the polar cone of the cone $C^{\mathcal{R}}$.

Infinity-dimensional optimal control problems

Let us consider the following optimal control problem

$$\varphi(x(T)) \to \min \tag{3}$$

subject to the semilinear dynamics:

$$\begin{aligned} \dot{x}(t) = &\quad Ax(t) + f(t, x(t), u(t)) \text{ a.e. in } [0, T], \\ x(0) = &\quad x_0 \in X, \quad x(T) \in S \\ u(\cdot) \in &\quad \mathcal{U} := \{u(\cdot) : [0, T] \to U \mid u(\cdot) \text{ is strongly measurable}\}. \end{aligned} \tag{4}$$

Here X is a Banach space, S is a closed subset of X, U is a separable complete metric space, the linear operator A is the infinitesimal generator of a strongly continuous semigroup $\{\mathcal{K}(t) \mid t \geq 0\}$.

Infinity-dimensional optimal control problems

The function $f : [0, T] \times X \times U \to X$ is strongly measurable for any fixed $(x, u) \in Y \times U$, the function f is Fréchet differentiable in x for any fixed (t, u), the function φ is strictly Fréchet differentiable, the functions $f(t, \cdot, \cdot)$ and $f'_x(t, \cdot, \cdot)$ are jointly continuous, $f'_x(t, \cdot, u)$ is a locally uniformly continuous function uniformly with respect to $u \in U$ and $t \in [0, T]$. Moreover, there is $M > 0$ such that

$$\|f'_x(t, x, u)\| \leq M \quad \text{and} \quad \|f(t, 0, u)\| \leq M$$

for each $(t, x, u) \in [0, T] \times Y \times U$.

It is standard to consider the set $\mathcal{U}$ endowed with the metric

$$\text{dist}\,(u_1(\cdot), u_2(\cdot)) := \mathrm{meas}\,\{t \in [0, T] : \; u_1(t) \neq u_2(t)\}.$$

Then $(\mathcal{U}, \text{dist})$ is a complete metric space.

Infinity-dimensional optimal control problems

Remark.

A mild solution $x(\cdot; u)$ of (4) corresponding to the admissible control u is, by definition, a continuous solution of the integral equation

$$x(t; u) = \mathcal{K}(t)x_0 + \int_0^t \mathcal{K}(t-\sigma)f(\sigma, x(\sigma), u(\sigma))\, d\sigma, \;\; t \in [0, T].$$

Notations.

Let $\tilde{\mathbf{B}}$ denote the unit ball in $X \times R$ centered at the origin and $G(u) := x(T, u)$ denote the mild solution of the control system (4) corresponding to the control $u \in \mathcal{U}$ at the moment T. Let us fix $\bar{u} \in \mathcal{U}$ and set $\bar{x} := G(\bar{u})$.

Diffuse variations

The diffuse variations are introduced to our knowledge by Xunjing Li and his co-workers in the early 1980's: : Given a real number $\rho > 0$, and two elements u and v of $\mathcal{U}$, then there exists of a subset E_ρ of $[0, T]$ of measure ρT such that

$$\sup_{t\in[0,T]} \left| \int_0^t h(t,s)\, ds - \frac{1}{\rho} \int_0^t \chi_{E_\rho} h(t,s)\, ds \right| = o_\rho(1)$$

with

$$h(t,s) = \mathcal{K}(t-s)[f(s, x(s;u), v(s)) - f(s, x(s;u), u(s))].$$

Next, we define

$$u_{\rho,v}(t) = \begin{cases} v(t), & \text{if } t \in E_\rho, \\ u(t), & \text{otherwise.} \end{cases}$$

Infinity-dimensional optimal control problems

Li and Yong (1994) proved that for these two elements u and v of $\mathcal{U}$, the limit

$$\frac{d}{d\rho}x(T,u_{\rho,v})\Big|_{\rho=0}=\lim_{\rho\to 0+}\frac{1}{\rho}(x(T,u_{\rho,v})-x(T,u))$$

exists and equals $\xi(T,u,v)$, where $\xi(\cdot,u,v)$ is the mild solution of the equation

$$z'(t)=\Big(A+\partial_x f(t,x(t,u),u(t))\Big)z(t)+ \\ +\Big(f(t,x(t,u),v(t))-f(t,x(t,u),u(t))\Big)$$

with $z(0)=0$. Let us denote by

$$\mathcal{V}_D(\bar{u}):=\Big\{\xi(T,\bar{u},v):\ v\in\mathcal{U}\Big\}$$

the set of all state diffuse variations at the point $\bar{u}$.

Theorem.

Let us fix $\bar{u} \in \mathcal{U}$. Let C^S be an uniform tangent cone to the target set S at the point $\bar{x}$ and $C^{\mathcal{R}}$ be an uniform tangent cone to the reachable set $\mathcal{R} := \{G(u) : \ u \in \mathcal{U}\}$ at the point $\bar{x}$. We set

$$C^{\tilde{\mathcal{R}}} := \{(v, \varphi'(\bar{x})v) : \ v \in C^{\mathcal{R}}\} \text{ and } C^{\tilde{S}} := \{(v, r) : \ v \in C^S, \ r \leq 0\}.$$

(i) If there exists $\rho > 0$ such that the set

$$\mathrm{co}\left(\left(C^{\tilde{\mathcal{R}}} \cap \ \tilde{\mathbf{B}}\right) - \left(C^{\tilde{S}} \cap \ \tilde{\mathbf{B}}\right)\right)$$

is dense in $\rho\tilde{B}$, then $\bar{u}$ is not optimal;

Theorem. (contimuation)

(ii) If the cone $\operatorname{co}\left(C^{\tilde{\mathcal{R}}} - C^{\tilde{S}}\right)$ is not dense in $X \times R$ and $\overline{\operatorname{co}}\ C^{\mathcal{R}}$ contains all diffuse variations at $\bar{u}$, then the following necessary condition of Pontryagin maximum principle type holds true: there exist a nontrivial

$$(\psi(\cdot), \psi^0) \in C_{w^*}([0, T], X^*) \times (-\infty, 0] \text{ such that}$$

$$\psi(t) = \mathcal{K}(t - T)\psi(T) + \int_t^T \mathcal{K}(t - s)\left(f_x'(s, \bar{x}(s), \bar{u}(s))\right)^* \psi(s)\, ds,$$

$$H(t, \bar{x}(t), \bar{u}(t), \psi(t)) = \max_{u \in U} H(t, \bar{x}(t), u, \psi(t)) \text{ a.e. in } [0, T],$$

$$\left\langle \psi(T) + \psi^0 \varphi'(\bar{x}(T)), v \right\rangle \geq 0 \quad \text{for each } v \in C^S$$

and $\langle \psi(T), v \rangle \leq 0$ for each $v \in C^{\mathcal{R}}$, where

$$H(t, y, u, \psi) = \langle \psi, f(t, y, u) \rangle .$$

Infinity-dimensional optimal control problems

Corollary

Let $\bar{u} \in \mathcal{U}$ be a solution of the optimal control problem (3)–(4) and let $\bar{x} := G(\bar{u})$. Let C^S be an uniform tangent cone to the target set S at the point $\bar{x}$ and $C^{\mathcal{R}}$ be an uniform tangent cone to the reachable set $\mathcal{R} := \{G(u) : \ u \in U\}$ at the point $\bar{x}$. We set

$$C^{\tilde{\mathcal{R}}} := \{(v, \varphi'(\bar{x})v) : \ v \in C^{\mathcal{R}}\} \text{ and } C^{\tilde{S}} := \{(v, r) : \ v \in C^S, \ r \leq 0\}.$$

If the set

$$\left(C^{\tilde{\mathcal{R}}} \cap \ \tilde{\mathbf{B}}\right) - \left(C^{\tilde{S}} \cap \ \tilde{\mathbf{B}}\right)$$

is quasisolid and $\overline{\text{co}}\ C^{\mathcal{R}}$ contains the set of all diffuse variations at $\bar{u}$, then the following necessary condition of Pontryagin maximum principle type holds true:

Infinity-dimensional optimal control problems

Corollary (continuation).

there exist a nontrivial $(\psi(\cdot), \psi^0) \in C_{w^*}([0, T], X^*) \times (-\infty, 0]$ such that

$$\psi(t) = \mathcal{K}(t - T)\psi(T) + \int_t^T \mathcal{K}(t - s) \left(f'_x(s, \bar{x}(s), \bar{u}(s))\right)^* \psi(s)\, ds,$$

$$H(t, \bar{x}(t), \bar{u}(t), \psi(t)) = \max_{u \in U} H(t, \bar{x}(t), u, \psi(t)) \text{ a.e. in } [0, T],$$

$$\left\langle \psi(T) + \psi^0 \varphi'(\bar{x}(T)), v \right\rangle \geq 0 \quad \text{for each } v \in C^S$$

and

$$\langle \psi(T), v \rangle \leq 0 \quad \text{for each } v \in C^{\mathcal{R}},$$

where

$$H(t, y, u, \psi) = \langle \psi, f(t, y, u) \rangle .$$

Infinity-dimensional optimal control problems

Remark.

The above written corollary remains valid if one replaces $C^{\tilde{\mathcal{R}}} \cap \tilde{\mathbf{B}}$ (resp. $C^{\tilde{S}} \cap \tilde{\mathbf{B}}$) by the set of all diffuse variations (resp. by a uniform tangent set to $\tilde{S}$ at $(\bar{x}, \varphi(\bar{x}))$). In particular, if the target set S is convex one can obtain the following

Corollary.

Let $\bar{u} \in \mathcal{U}$ be a solution of the optimal control problem (3)–(4) and let $\bar{x} := G(\bar{u})$. Let the target set S be convex and $\mathcal{V}_D(\bar{u})$ be the set of all diffuse variations at the point $\bar{u}$. If the set

$$\mathcal{V}_D(\bar{u}) - (S - \bar{x})$$

is quasisolid, then the following necessary condition of Pontryagin maximum principle type holds true:

Infinity-dimensional optimal control problems

Corollary (continuation).

there exist a nontrivial $(\psi(\cdot), \psi^0) \in C_{w^*}([0, T], X^*) \times (-\infty, 0]$ such that

$$\psi(t) = \mathcal{K}(t-T)\psi(T) + \int_t^T \mathcal{K}(t-s)\left(f'_x(s, \bar{x}(s), \bar{u}(s))\right)^* \psi(s)\, ds,$$

$$H(t, \bar{x}(t), \bar{u}(t), \psi(t)) = \max_{u \in U} H(t, \bar{x}(t), u, \psi(t)) \text{ a.e. in } [0, T],$$

$$\left\langle \psi(T) + \psi^0 \varphi'(\bar{x}(T)), v \right\rangle \geq 0 \quad \text{for each } v \in S - \bar{x}$$

and

$$\langle \psi(T), v \rangle \leq 0 \quad \text{for each } v \in \mathcal{V}_D(\bar{u}),$$

where

$$H(t, y, u, \psi) = \langle \psi, f(t, y, u) \rangle .$$

A functional analytic approach to a Bolza problem

A natural next step is to apply the obtained abstract Lagrange multiplier rule to optimal control problems in finite-dimensional state space considered as optimization problems on an infinite-dimensional space of the corresponding trajectories.

There exist very general nonsmooth necessary optimality conditions (cf., e.g., books by F.Clarke, A.Ioffe) for such problems. We considered the most simple, but non trivial control problem: the so called problem of the calculus of variations. We proved a result which is very close to Theorem 18.1 of F. Clarke, Functional Analysis, Calculus of Variations and Optimal Control. Our approach is completely different from the techniques presented there. We do not use any variational principles. Our proofs heavily rely on the specific functional-analytic properties of the considered problem.

A functional analytic approach to a Bolza problem

We study the classical

Problem of the calculus of variations

$$\varphi(x) = \int_a^b L(x(t), \dot{x}(t))\, dt \to \min \ \text{ subject to } x(a) = x_a \text{ and } x(b) = x_b,$$

where $x : [a, b] \to R^n$ is an absolutely continuous curve.

This classical problem is considered under different assumptions imposed on the integrand L and various necessary conditions are obtained. Here we are going to consider the continuous case, i.e. we assume that $L : R^n \times R^n \to R$ is continuous.

A functional analytic approach to a Bolza problem

Let X be the Banach space $L^1([a,b];R^n)$, $Y := X \times X \equiv L^1([a,b];R^{2n})$. Then $X^* = L^\infty([a,b];R^n)$ and $Y^* = X^* \times X^* \equiv L^\infty([a,b];R^{2n})$. We consider the integral functional $\varphi : Y^* \equiv X^* \times X^* \to R$ defined by

$$\varphi(y) = \int_a^b L(y(t))\, dt.$$

We set

$$P := \left\{ (x,u) \in X^* \times X^* : \ x(t) = x_a + \int_a^t u(s)\, ds, \ t \in [a,b] \right\}$$

and

$$Q := \left\{ (x,u) \in X^* \times X^* : \ \int_a^b u(s)\, ds = x_b - x_a \right\},$$

where x_a and x_b are fixed points of R^n.

A functional analytic approach to a Bolza problem

We now consider the

Variational problem

(VP) $\varphi(x, u) \to \min$ subject to $(x, u) \in P \cap Q$.

To obtain a necessary optimality condition based on the abstract Lagrange multiplier rule we need:
- to construct a uniform tangent cone C to Epi φ;
- to construct a uniform tangent cone A to the target $P \cap Q$;
- quasi-solidity of the difference $C \cap \bar{\mathbf{B}}_{Y^* \times \mathbb{R}} - (A \times (-\infty, 0]) \cap \bar{\mathbf{B}}_{Y^* \times \mathbb{R}}$;
- to prove regularity of the obtained Lagrange multipliers.

Definition.

If A is a subset of Y and B, the polar of A is the set $A^0 := \{y^* \in Y^* : \ z^*(z) \leq 1 \text{ for every } y \in A\}$ and the prepolar of B is the set $B_0 := \{y \in Y : \ y^*(z) \leq 1 \text{ for every } y^* \in B\}$.

A functional analytic approach to a Bolza problem

Let $(\bar{x}, \bar{u}) \in X^* \times X^*$ be a solution of the problem **(VP)** and let $A := P \cap Q - (\bar{x}, \bar{u})$. Clearly, A is an uniform tangent cone to $P \cap Q$ at $(\bar{x}, \bar{u})$.

du Bois-Raymond lemma

The set A is a w^*-closed linear subspace of $X^* \times X^*$ and its pre-polar is the set

$$A_0 = \{(y, v) \in X \times X : v \text{ absolutely continuous}, \ \dot{v}(t) = y(t) \text{ a.e. in [a,b]}\}.$$

A functional analytic approach to a Bolza problem

Next we are going to construct a uniform tangent cone to the epigraph

$$\text{Epi }(\varphi) := \{(y,r) \in Y^* \times R : \ \varphi(y) \le r, \ y \in Y\}$$

at the point $(\bar{y}, \varphi(\bar{y}))$.
Let $\hat{T}_{epi\ L}(\bar{y}(t), L(\bar{y}(t)))$ denote the Clarke tangent cone to the epigraph *epi* $L \subset R^{2n+1}$ of L at the point $(\bar{y}(t), L(\bar{y}(t)))$. Let $R : [a,b] \longrightarrow R$ be a nonnegative summable function. We are going to assume that $R \ge 1$ on $[a,b]$. We introduce the multivalued map

$$t \to G_R(t) := \hat{T}_{epi\ L}(\bar{y}(t), L(\bar{y}(t))) \cap \left(\bar{\mathbf{B}}_{\mathbb{R}^{2n}} \times [-R(t), R(t)]\right), \quad t \in [a,b].$$

Lemma

The multivalued map $G_R : [a,b] \longrightarrow R^{2n+1}$ is measurable.

A functional analytic approach to a Bolza problem

It is natural to to try to use the uniformity of the Clarke tangent cone in finite dimensional space. We set

$$B_R := \left\{ (v, r) \in Y^* \times R : \begin{array}{c} \text{there exists a measurable function } r_v \text{ with} \\ r = \displaystyle\int_a^b r_v(t)dt \text{ and for almost all } t \in [a, b] \\ \text{it is true that } (v(t), r_v(t)) \in G_R(t) \end{array} \right\}$$

Let C be the cone generated by B_R, i.e. $C = \{\lambda y : \ y \in B_R, \ \lambda \geq 0\}$. To obtain that B_R is a uniform tangent set to Epi φ at $(\bar{y}, \varphi(\bar{y}))$ we need some additional assumption except for measurability of R.

A functional analytic approach to a Bolza problem

Definition

It is said that the set $\mathcal{F}$ of summable functions $f : [a, b] \to R^n$ is uniformly integrable if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for every measurable subset E of $[a, b]$ with meas $(E) < \delta$ it holds true $\int_E |f(t)|dt < \varepsilon$ for every $f \in \mathcal{F}$.

Standing assumption (SA)

There exist a positive real $\bar{\delta}$ such that the family of summable functions

$$\mathcal{F} := \left\{ \frac{L(y + \lambda v) - L(y)}{\lambda} - r_v : \begin{array}{l} y \in Y^* \text{ with } \|y - \bar{y}\| < \bar{\delta}, \lambda \in (0, \bar{\delta}), \\ (v, r) \in B_R \text{ with } r = \displaystyle\int_a^b r_v(t)dt \\ \text{and for almost all } t \in [a, b] \\ \text{ it is true that } (v(t), r_v(t)) \in G_R(t) \end{array} \right\}$$

is uniformly integrable.

Lemma

If the condition **(SA)** *holds true, then the set B_R is an uniform tangent set to Epi (φ) at the point $\bar{y}$.*

We say that the vector $\xi \in R^{2n}$ belongs to the Clarke subdifferential $\partial_C L(z)$ of L at the point $z \in R^{2n}$ iff $(\xi, -1)$ belongs to the Clarke tangent cone to $(z, L(z))$, i.e.

$$\xi \in \partial_C L(z) \text{ iff } (\xi, -1) \in \hat{T}_{epi\ L}(z, L(z)).$$

A functional analytic approach to a Bolza problem

Let us introduce the set of all summable selections of the Clarke subdifferential of L along the optimal couple $\bar{y}(t)$, $t \in [a, b]$, i.e.

$$K := \{w \in Y : \; w(t) \in \partial_C L(\bar{y}(t)) \text{ a.e. in } [a, b]\} .$$

Lemma

Let the set $\partial_C L(\bar{y}(t))$ be nonempty, bounded for almost every $t \in [a, b]$ and let all its measurable selections be summable. Then the prepolar C_0 of the cone C coincides with the set

$$\tilde{K} := \{\lambda(w, -1) : \; \lambda \geq 0, \; w \in K\}.$$

Lemma

Let us assume that $\partial_C L(\bar{y}(t)) \subset R(t)\bar{\mathbf{B}}_{\mathbb{R}^n}$ *for almost every* $t \in [a, b]$. *Then* C *is weak star closed and* C *coincides with the polar of* $\tilde{K}$, *i.e.* $C = \tilde{K}^0$. *Moreover, if the standing assumption (SA) holds true, then* C *is an uniform tangent cone to* $Epi\ \varphi$.

Remark.

The proof is based on the Banach-Dieudonné theorem.

The next lemma is crucial for obtaining regular separating functional (belonging to Y, not to Y^{**}) and it relies on the fact that weak compact sets are weak star closed when embedded in the second dual.

Lemma

Let us assume that $\partial_C L(\bar{y}(t)) \subset R(t)\bar{\mathbf{B}}_{\mathbb{R}^n}$ for almost every $t \in [a, b]$. Then C has nonempty interior and the polar C^0 of C coincides with the embedding image $J(C_0)$ of its prepolar, and hence coincides with the embedding image $J(\tilde{K})$ of $\tilde{K}$, i.e.

$$C^0 = J(C_0) = J(\tilde{K}),$$

where J is the canonical embedding of $Y \times R$ in $Y^{**} \times R$.

A functional analytic approach to a Bolza problem

Theorem

*Let $L : R^n \times R^n \longrightarrow R$ be a continuous mapping. Let $\bar{y} = (\bar{x}, \bar{u})$ be a solution of **(VP)**. Let us assume that there exists a nonnegative summable function R on $[a, b]$ such that $\partial_C L(\bar{y}(t)) \subset R(t)\bar{\mathbf{B}}_{\mathbb{R}^n}$ for almost every $t \in [a, b]$. Let the standing assumption (SA) holds true. Then there exists an absolutely continuous function $p : [a, b] \longrightarrow R^n$ such that*

$$(\dot{p}(t), p(t)) \in \partial_C L(\bar{y}(t)) \text{ a.e. in } [a, b] \ .$$

Various types of control variations

Let us consider the following control system Σ:

$$\begin{aligned} \dot{x}(t) = &\ Ax(t) + f(t, x(t), u(t)) \text{ a.e. in } [0, T], \\ x(0) = &\ x_0 \in X, \\ u(\cdot) \in &\ \mathcal{U} := \{u(\cdot) : [0, T] \to U \mid u(\cdot) \text{ is strongly measurable}\}. \end{aligned}$$

Here X is a Banach space, U is a complete metric space, the linear operator A is the infinitesimal generator of a strongly continuous semigroup $\{S(t) \mid t \geq 0\}$. It is standard to consider the set $\mathcal{U}$ endowed with the metric

$$\text{dist}\,(u_1(\cdot), u_2(\cdot)) := \text{meas}\,\{t \in [0, T] : \ u_1(t) \neq u_2(t)\}.$$

It turns out that $(\mathcal{U}, \text{dist})$ is a complete metric space (cf. Proposition 3.10 on p. 145 in [Li, Yong, Optimal control theory for infinite dimensional systems, 1994]).

Various types of control variations

Solutions of Σ

A solution of Σ corresponding to the control $u \in \mathcal{U}$ is a continuous solution of the integral equation

$$x(t,u) = S(t)x_0 + \int_0^t S(t-\sigma)f(\sigma, x(\sigma), u(\sigma))\, d\sigma, \;\; t \in [0, T].$$

Various types of control variations

Abstract first-order variations $\mathcal{V}(\bar{u})$ (H. Frankowska (1990))

Let $\bar{u} \in \mathcal{U}$. We denote by $B_h(\bar{u})$ (resp. $\overline{B}_h(\bar{u})$) the open (resp. closed) ball centered at $\bar{u}$ with radius h. We set

$$\mathcal{V}(\bar{u}) := \limsup_{h \to 0+} \frac{G(B_h(\bar{u})) - G(\bar{u})}{h},$$

where $G(B_h(\bar{u}))$ is the set of all trajectories' ends at $t = T$ for $u \in B_h(\bar{u})$ and lim sup is in the sense of Kuratowski.

In other words $v \in \mathcal{V}(\bar{u})$ if and only if there exist sequences $h_i \to 0+$ and $v_i \to v$ such that $G(\bar{u}) + h_i v_i \in G(B_{h_i}(\bar{u}))$. It is clear that $\mathcal{V}(\bar{u})$ is a closed subset of X.

Classical needle variations are proposed by McShane (1936) and used by L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mischenko (1961)

Given two real numbers $\tau \in (0, T]$, $\rho > 0$ with $\rho \le \tau$, $u \in \mathcal{U}$ and an element v of U, we define the classical needle control variation

$$u_{\tau,\rho,v}(t) = \begin{cases} v, & \text{if } t \in [\tau - \rho, \tau], \\ u(t), & \text{otherwise.} \end{cases}$$

Various types of control variations

Fattorini (1987) proved that for each $u \in \mathcal{U}$ there exists a set Ω_u of full measure in $[0, T]$ such that the limit

$$\frac{d}{d\rho} G(u_{\tau,\rho,v})\Big|_{\rho=0} = \lim_{\rho \to 0+} \frac{1}{\rho}\left(G(u_{\tau,\rho,v}) - G(u)\right)$$

exists and equals $\xi(T, \tau, u, v)$ for each $\tau \in \Omega_u$, where

$$\xi(t, s, u, v) = \Phi(t, s; u)(f(s, x(s, u), v) - f(s, x(s, u), u(s))),$$

with $\Phi(t, s; u)$ being the solution of the equation

$$z'(t) = \Big(A + \partial_x f(t, x(t, u), u(t))\Big) z(t), \;\; z(s) = \text{ Id},$$

where Id is the identity operator in X.

Various types of control variations

Here the strongly continuous operator $\Phi(t, s; u)$ obeys the integral equation

$$\Phi(t, s; u)y = S(t-s)y + \int_s^t S(t-\sigma)B(\sigma)\Phi(\sigma, s; u)y \, d\sigma, \; y \in X, \; s \leq t \leq T,$$

where $B(t) = \partial_x f(t, x(t, u), u(t))$.

Let us denote by

$$\mathcal{V}_N(\bar{u}) := \left\{\Phi(T,\tau;\bar{u})(f(\tau,x(\tau,\bar{u}),v) - f(\tau,x(\tau,\bar{u}),\bar{u}(\tau))) : \ v \in U, \tau \in \Omega_{\bar{u}}\right\}$$

the set of all state needle variations.

Standard arguments using packages of control needle variations yields the inclusion

$$\text{co}\ \mathcal{V}_N(\bar{u}) \subset \mathcal{V}(\bar{u}).$$

Various types of control variations

Diffuse control variations are introduced by Xunjing Li and his co-workers (1985)

Given a real number $\rho > 0$, and two elements u and v of $\mathcal{U}$, it can be proved in the existence of a subset E_ρ of $[0, T]$ of measure ρT such that

$$\sup_{t\in[0,T]} \left| \int_0^t h(t,s)\, ds - \frac{1}{\rho}\int_0^t \chi_{E_\rho} h(t,s)\, ds \right| = o_\rho(1)$$

with

$$h(t,s) = S(t-s)[f(s, x(s;u), v(s)) - f(s, x(s;u), u(s))].$$

Next, we define

$$u_{\rho,v}(t) = \begin{cases} v(t), & \text{if } t \in E_\rho, \\ u(t), & \text{otherwise.} \end{cases}$$

Various types of control variations

Li and Yong (1994) proved that for these two elements u and v of $\mathcal{U}$, the limit

$$\frac{d}{d\rho} G(u_{\rho,v})\Big|_{\rho=0} = \lim_{\rho \to 0+} \frac{1}{\rho}\left(G(u_{\rho,v}) - G(u)\right)$$

exists and equals $\xi(T, u, v)$, where $\xi(\cdot, u, v)$ is the solution of the equation

$$z'(t) = \Big(A + \partial_x f(t, x(t,u), u(t))\Big) z(t) + \Big(f(t, x(t,u), v(t)) - f(t, x(t,u), u(t))$$

with $z(0) = 0$. Let us denote by

$$\mathcal{V}_D(\bar{u}) := \Big\{\xi(T, \bar{u}, v) : \ v \in \mathcal{U}\Big\}$$

the set of all state diffuse variations.

Various types of control variations

Li and Chow (1987) developed in a new technique for convexification of these variations. In particular they proved that

$$\text{co } \mathcal{V}_D(\bar{u}) \subset \mathcal{V}(\bar{u}).$$

H. Frankowska (1990) introduced yet another set of state variations:

$$\mathcal{V}_{Fr}(\bar{u}) := \Big\{ \int_0^T \Phi(T, t; \bar{u}) y(t) \; dt :$$

$$y(t) \in \overline{\text{co}}\Big(f(t, x(t, \bar{u}), U) - f(t, x(t, \bar{u}), \bar{u}(t)) \Big),$$

$$y(\cdot) \text{ is strongly measurable} \Big\}.$$

Various types of control variations

A corollary of a result of Frankowska (1990)

Let

$$\overline{\text{co}}\, \mathcal{V}_N(\bar{u}) \supset \rho \bar{B}$$

for some $\rho > 0$, where $\bar{B}$ is the closed unit ball. Then there exists $\bar{\delta} > 0$ such that

$$\overline{\text{co}}\, \mathcal{V}_N(u) \supset \frac{\rho}{2} \bar{B}$$

for each $u \in B_{\bar{\delta}}(\bar{u})$.

Important!

The same property have the convex closed hull of the set of all diffuse variations as well as the set of all variations introduced by Frankowska.

Various types of control variations

Proposition 1.

$$\operatorname{co} \mathcal{V}_D(\bar{u}) \subset \mathcal{V}_{Fr}(\bar{u}) \subset \overline{\operatorname{co}}\, \mathcal{V}_N(\bar{u}) \subset \mathcal{V}(\bar{u}).$$

It is natural to expect that $\mathcal{V}_N(\bar{u}) \subset \mathcal{V}_D(\bar{u})$.

But the next example shows that this hypothesis is false !!!
Moreover, in this example, the set $\mathcal{V}_{Fr}(\bar{u})$ has empty interior, while co $\mathcal{V}_N(\bar{u})$ contains an open ball centered at the origin!!!

Example

We set $X \equiv l_1$ and let

$$e_1, e_2, \ldots, e_n, \ldots$$

be its standard orthogonal basis of unit vectors. If

$$t_0 := 0, t_1 := \frac{1}{2}, \ldots, t_n := \frac{1}{2^n}, \ldots,$$

then we define the following function

$$g(t) := \begin{cases} -e_i + 2\dfrac{t - t_{i-1}}{t_i - t_{i-1}} e_i, & \text{for } t \in [t_{i-1}, t_i], \\ 0, & \text{for } t = 1. \end{cases}$$

Example (continuation)

Because l_1 is separable and the preimage of $g^{-1}(U)\bigcap(t_{i-1}, t_i)$ is open for each open interval U and for each positive integer i, the set $g^{-1}(U)$ is Borel measurable. Clearly,

$$\int_0^1 \|g(t)\| \, dt \leq \int_0^1 1 \, dt = 1.$$

We set $U = \left\{u_0, u_1\right\}$, $f(t,u) := \begin{cases} 0, & \text{for } u = u_0; \\ g(t), & \text{for } u = u_1. \end{cases}$ and consider the following control system:

$$\dot{x}(t) = f(t, u(t)), \; x(0) = 0, \; \text{and } u \in \mathcal{U},$$

where $\mathcal{U} = \left\{u : [0,1] \to U : \; u \text{ is measurable}\right\}$.

Example (continuation)

Clearly

$$x_u(1) = \int_0^1 f(t, u(t))\, dt \quad \text{for each } u \in \mathcal{U}.$$

We set $\bar{u} \equiv u_0$. For this example $\Phi(T, t; \bar{u}) = \mathrm{Id}$. Then we have

$$\mathcal{V}_{Fr}(\bar{u}) = \left\{ \int_0^1 v(t)\, dt : \ v(t) \text{ is a measurable selection of } \overline{\mathrm{co}}\Big(\{g(t)\} \cup \{0\}\Big) \right\}$$

$$\subset \left\{ \int_0^1 v(t)\, dt : \ v(t) \in [-e_i, e_i) \ \text{ for } \ t \in [t_{i-1}, t_i), \ i = 1, 2, \dots \right\}.$$

The countable additivity of Bochner integral implies that

$$\int_0^1 v(t)\, dt = \sum_{i=1}^{\infty} \int_{t_{i-1}}^{t_i} v(t)\, dt.$$

Example (continuation)

On the other hand each term of the above written sum is contained in the closed convex hull of the values of v (belonging to line segment $[-e_i, e_i]$) multiplied by the length of the interval $[t_{i-1}, t_i]$, that is $1/2^i$. Hence

$$\int_{t_{i-1}}^{t_i} v(t)\, dt \in \left[-\frac{e_i}{2^i}, \frac{e_i}{2^i}\right]$$

and

$$\mathcal{V}_{Fr}(\bar{u}) \subset A := \left\{x = (x_1, x_2, \dots, x_n, \dots) \in l^1 : \; x_i \in \left[-\frac{1}{2^i}, \frac{1}{2^i}\right]\right\}.$$

$$\text{for} \quad i = 1, 2, \dots$$

Clearly, the set A is a closed convex subset of l^1 with empty interior. On the other hand, the closed convex hull of the set of needle variations coincides with the unit ball of l^1.

An abstract consideration

Abstract first-order variations: Frankowska (1990)

Consider a complete metric space $\mathcal{U}$ and a Banach space X. Let $G : \mathcal{U} \to X$ be a single-valued continuous map. We say that the contingent variation of G at the point u is the closed subset of X

$$G^{(1)}(u) := \limsup_{h \to 0^+} \frac{G(B_h(u)) - G(u)}{h},$$

i.e. $v \in G^{(1)}(u)$ iff there exists sequences $h_i \to 0^+$ and $v_i \to v$ such that $G(u) + h_i v_i \in G(B_{h_i}(u))$.

An abstract consideration

Abstract first-order variations: Frankowska (1990)

Note that the set $G^{(1)}(u)$ is star shaped, i.e. if $v \in G^{(1)}(u)$ and $0 < \alpha < 1$ then $\alpha v \in G^{(1)}(u)$. Indeed, the inclusion $v \in G^{(1)}(u)$ implies the existence of a sequence $v_i \to v$ and $h_i \to 0^+$ as $i \to \infty$ such that $G(u) + h_i v_i \in G\left(B_{h_i}(u)\right)$. Then $\alpha v \in G^{(1)}(u)$ because $\alpha v_i \to \alpha v$, $\dfrac{h_i}{\alpha} \to 0^+$ and

$$G(u) + \frac{h_i}{\alpha}\,\alpha v_i = G(u) + h_i v_i \in G\left(B_{h_i}(u)\right) \subset G\left(B_{\frac{h_i}{\alpha}}(u)\right).$$

Regardless of the generality of the above written definition of the contingent variation, we always have in mind $G(u)$ to be the solution of the control system Σ corresponding to the control $u \in \mathcal{U}$ at the moment $t = T$. Recall that $(\mathcal{U}, \mathrm{dist})$ is a complete metric space and $G(\cdot)$ is continuous on it. Moreover, in this case $G^{(1)}(u) = \mathcal{V}(u)$.

An abstract consideration

Bouligand's tangent cone

Let X be a Banach space. It is said that an element v of X belongs to the Bouligand's tangent cone $T_S(x)$ to S at the point x iff there exist a sequence v_i in X converging to v and a sequence t_i in $(0, +\infty)$ decreasing to 0 such that $x + t_i v_i \in S$ for all indices i.

Clarke's tangent cone

Let X be a Banach space. It is said that an element v of X belongs to the Clarke's tangent cone $\hat{T}_S(x)$ to S at the point x iff, for every sequence x_i in S converging to x and and for any sequence t_i in $(0, +\infty)$ decreasing to 0, there exists a sequence v_i in X converging to v such that $x_i + t_i v_i \in S$ for all indices i.

An abstract consideration

Control variations of Bouligand type

It is said that v is a control variation of Bouligand type at $\bar{u}$ if there exist sequences $h_i \to +0$, $v_i \to v$ and $\bar{u}_i \to \bar{u}$ such that $G(\bar{u}_i) = G(\bar{u}) + h_i v_i$.

We denote by $T_G(\bar{u})$ the set of all control variations of Bouligand type at $\bar{u}$.

Control variations of Clarke type

It is said that v is a control variation of Clarke type if for every sequences $h_i \to +0$ and $u_i \to \bar{u}$ there exist sequences $v_i \to v$ and $\bar{u}_i \to \bar{u}$ such that $G(\bar{u}_i) = G(u_i) + h_i v_i$.

We denote by $\hat{T}_G(\bar{u})$ the set of all control variations of Clarke type at $\bar{u}$.

An abstract consideration

Proposition

The following assertions hold true:

i) If v is a control–state variation of Clarke type at $\bar{u}$ and w is a control–state variation of Bouligand type at $\bar{u}$ and $\lambda \in [0,1]$, then $\lambda v + (1-\lambda)w$ is a control–state variation of Bouligand type at $\bar{u}$;

ii) If v and w are a control variations of Clarke type and $\lambda \in [0,1]$, then $\lambda v + (1-\lambda)w$ is also a control variation of Clarke type.

iii) Each state needle variation at $\bar{u}$ is a control-state variation of Clarke type at $\bar{u}$.

Remark

A. Bressan proved in 1985 in a different way the assertion i) for the case $X = R^n$. R.-M. Bianchini and M. Kawski showed in 2002 the existence of high-order variations that can not be summed even in R^n.