The Maximum Principle and Symmetries: Quantum Control

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with a special session to honor Gianna Stefani

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Overview of the talk:

- The talk is principally motivated by the quantum control systems introduced by R. Brockett, N. Khaneja, S. Glazer, J.P. Gauthier, U. Boscain, B. Bonnard, T. Chambrion.
- Along the way we will talk about the symmetries that are pertinent for the solutions of this class of problems.
- We will also talk briefly about the relevance of the classical theory of symmetric Riemannian spaces for the solutions of quantum systems.
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Some relevant references:

1. Boscain, U., Chambrion, T., Gauthier, J.-P., On the K+P problem for a three-level quantum system: optimality implies resonance, J. Dynam. Control Systems 8, no. 4, 2002.
2. R. Brockett, S. Glazer, and N.Khaneja, Sub-Riemannian geometry and time optimal control of three spin systems: quantum gates and coherence transfer, Phys. Rev. A (3) 65), no. 3, part A, 032301, 2002.
3. Khaneja N., Yuan H., Zeier R., Elliptic functions and efficient control of Ising spin chains with unequal couplings, Phys. Rev. A, 77, 032340, 2008.
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6. V. Jurdjevic Optimal Control and Geometry: Integrable Systems Cambridge studies in Advanced Mathematics 154, Cambridge

Introduction;

- Bilinear systems: $\frac{dx}{dt} = (A + \sum_{i=1}^{m} B_i u_i(t))(x(t)),\ x(t) \in R^n$.
- Does this systems admit any symmetries?
- The answer necessarily involves the fundamental matrix system:

$$\frac{dg}{dt} = (A + \sum_{i=1}^{m} B_i u_i(t))g(t),\ g(t) \in Gl_n(R), \\ x(t) = g(t)x_0. \tag{1}$$

- (1) is right-invariant: if $g(t)$ is a solution so is $g(t)g_0$.
- $x(t) = g^{-1}(t)x_0$ leads to

$$\frac{dg}{dt} = -g(t)(A + \sum_{i=1}^{m} B_i u_i(t)) \tag{2}$$

- $\frac{dg^{-1}}{dt} g = -g^{-1}\frac{dg}{dt}$.
- Any energy $\int F(u(t))dt$ is invariant under either left or right translations.

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Lie algebra constraints

- Matrices $A, B_1, \ldots B_m$ usually have additional structure: for example, trace zero, skew symmetry, symplectic structure, etc.
- $g(t) \in G$, where G is the group whose Lie algebra $\mathfrak{g} = Lie(A_1, B_1, \ldots, B_m)$.
- Bilinear systems are the prototype of the following general situation: G acts (either from right or left) on a manifold M. Then each element $A \in \mathfrak{g}$ defines a vector field X_A on M: $(g, x) \to \phi_g(x)$, then $X_A(x) = \frac{d}{dt}\phi_{e^{At}}(x)|_{t=0}$.
- $$\frac{dg}{dt} = (A + \sum_{i=1}^{m} u_i(t)B_i)g,$$ $$\frac{dx}{dt} = X_A(x) + \sum_{i=1}^{m} u_i(t)X_{B_i}(x)$$
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Quantum control

- Schroedinger's equation: $i\frac{\partial\psi}{\partial t}(r,t) = H\psi(r,t)$, $\psi(r,t)$ is a complex wave function, an element of a complex Hilbert space $\mathcal{H} = L^2(\Omega)$, and H Hermitian operator on $\mathcal{H}$.
- Finite dimensional version: $\mathcal{H}^n$ is an n-dimensional complex Hilbert space and H is a Hermitian linear operator on $\mathcal{H}^n$

$$\frac{dz}{dt} = -iHz, z \in \mathcal{H}^n. \qquad (3)$$

- In an orthonormal basis in $\mathcal{H}^n$, then $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, and $\langle z, w\rangle = \sum_{i=1}^n z_i \bar{w}_i$. Hermitian means $H^\dagger = \bar{H}^T = H$, where $\bar{H}$ is the complex conjugate, and H^T is the matrix transpose.
- If H is Hermitian then iH is skew Hermitian, that is, $(iH)^\dagger = -iH$.
- $g(t)^\dagger = (e^{-(iHt)})^\dagger = e^{(-iHt)^\dagger} = e^{(iHt)} = g^{-1}(t)$
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- It follows that $\langle gz, gw \rangle = \langle z, w \rangle, g \in U_n$. In particular, $||gz|| = ||z||$ and U_n is compact.
- $|Det(g)|^2 = Det(g)\bar{D}et(g) = (Det(gg^\dagger) = Det(I) = 1$. $Det(g) \in \{z \in \mathbb{C} : ||z|| = 1\}$.
- Special unitary group $SU_n = \{g \in U_n : Det(g) = 1\}$.
- Controlled Schroedinger: $\frac{dz}{dt} = (H_d + \sum_{i=1}^m u_i(t)H_i)z(t), z \in \mathbb{C}^n$, where $H_d, H_1, \ldots, H_m$ are given skew-Hermitian matrices.
- Master equation in U_n:

$$\frac{dg}{dt} = (H_d + \sum_{i=1}^{m} u_i(t)H_i)g(t), g(0) = I, g(t) \in U_n.$$

- Time Optimal problem (Brockett et al.), Minimum Energy $\frac{1}{2}\int_0^T \sum_{i=1}^m u_i^2(t)\,dt, (J.P.Gauthier, U.Boscain)$.
- Basic building block- Qubit: Two state quantum system represented by two complex numbers (z_1, z_2) such that $|z_1|^2 + |z_2|^2 = 1$.

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- $\mathcal{A}(e, \leq T)$ stands for the reachable set in T or less units of time from the group identity e.
- Time optimal function $T^*(g) = Inf\{T : g \in \mathcal{A}(e, \leq T\}, g \in G$.
- Relaxed time optimal function $T^*(g) = Inf\{T : g \in \bar{\mathcal{A}}(e, \leq T\}$, $g \in G$.
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- The Lie saturate of a control system $\frac{dx}{dt} = F(x, u)$ on a manifold M is the largest family of vector fields $LS(\mathcal{F})$ in $Lie(\mathcal{F})$ such that $cl(\mathcal{A}_{\mathcal{F}}(x)) = cl(\mathcal{A}_{LS(\mathcal{F})}(x)), x \in M$.
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- Proposition 1. Suppose $\mathcal{F} = \{X_0 + \sum_{i=1}^m u_i X_i, u_i \in R\}$. Then $\lambda X_i, i \geq 1, \lambda \in \mathbb{R}$ is in the strong Lie saturate of $\mathcal{F}$ and so is the Lie algebra generated by $X_1, \dots, X_m$.
- In the case of our quantum system $\frac{dg}{dt} = (H_d + \sum_{i=1}^m u_i(t) H_i) g(t), g(0) = I, g(t) \in U_n$, the above implies that the relaxed time optimal function is unchanged if the system is enlarged to $\frac{dg}{dt} = (H_d + U(t)) g(t), U(t) \in \mathfrak{k}$ where $\mathfrak{k}$ denotes the Lie algebra generated by the control matrices $H_1, \dots, H_m$.

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Geometric setting- Assumptions

- We will assume that all systems are defined by matrices in a semi-simple Lie algebra $\mathfrak{g}$ (principally in $\mathfrak{g} = su_n$) .
- We will assume that $\mathfrak{k}$ is a compact Lie subalgebra of $\mathfrak{g}$, that is, assume that K the subgroup generated by the exponentials in $\mathfrak{k}$ is compact. In SU_n this condition is automatically satisfied.
- We will assume that there exists a vector space $\mathfrak{p}$ that is orthogonal to $\mathfrak{k}$ relative to the Killing form that satisfies $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$, $[\mathfrak{p}, \mathfrak{k}] \subseteq \mathfrak{p}$, $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$ (Cartan decomposition).
- Condition $[\mathfrak{p}, \mathfrak{k}] \subseteq \mathfrak{p}$ implies that $\mathfrak{p}$ is an invariant subspace for each $Ad_h, h \in K$. Notation: $Ad_K^{\mathfrak{p}} = \{Ad_h|\mathfrak{p}, h \in K\}$.
- (G, K) is called irreducible symmetric pair if $Ad_K^{\mathfrak{p}}$ acts irreducibly on $\mathfrak{p}$. In this case $M = G/K$ is called Riemannian globally irreducible symmetric space.

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- The metric on $M = G/K$ is induced by the trace form $\langle X, Y \rangle = -\frac{1}{2}Trace(XY)$ which is Ad_K invariant.
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  $dim(\mathfrak{k}) = p^2 + q^2 - 1$, $dim(\mathfrak{p}) = 2pq$, $dim(M) = 2pq$.
- $SU_{n+1}/S(U_1 \times U_n) = \mathbb{C}P^n$.
- Regular elements. $A \in \mathfrak{p}$ is regular if the cenralizer of A in $\mathfrak{p}$ is an abelian subalgebra of $\mathfrak{g}$, i.e., if $\{X \in \mathfrak{g} : [X, A] = 0\} \cap \mathfrak{p}$ is abelian.
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- Master equation: $\frac{dg}{dt} = (A + U(t))g(t), g(t) \in SU_n, U(t) \in \mathfrak{k}$.
- Vertical system : $\frac{dh}{dt} = -h(t)U(t), h(t) \in K$.
- Horizontal system : $\frac{d\tilde{g}}{dt} = (h(t)Ah^{-1}(t))\tilde{g}(t), \tilde{g}(t) \in SU_n$.
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- 1. Characterize the time optimal solution of the convexified horizontal system.
  2. What are the solutions that transfer K to a terminal left coset Kg_1 in the time optimal fashion? (Brockett's question)
- The solutions of the Master equation and the solutions of the horizontal system (generated by the same control $U(t)$) project onto the same curve in G/K ($\tilde{g}(t) = h(t)g(t)$).
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Right-invariant Hamiltonians

- To take advantage of right-invariant symmetries trivialize $T^*G = G \times \mathfrak{g}^*$ by the right translations.
- Identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via the trace form: $\ell \in \mathfrak{g}^* \Leftrightarrow L \in \mathfrak{g}$ iff $\langle L, X \rangle = \ell(X)$.
- If H is any right invariant Hamiltonian, i.e., a function on $\mathfrak{g}^*$, then its Hamiltonian equations are

$$\frac{dg}{dt} = (dH)g, \ \frac{dL}{dt} = [dH, L].$$

- $L = L_{\mathfrak{p}} + L_{\mathfrak{k}}$, $dH = dH_{\mathfrak{p}} + dH_{\mathfrak{k}}$, then $\frac{dL}{dt} = [dH, L]$ becomes

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Extremal equations

- Hamiltonian lift of the horizontal system: $h_U = -\lambda + \langle L_{\mathfrak{p}}, hAh^{-1}\rangle$
- K is compact, hence $Max_{h\in K}\langle L_{\mathfrak{p}}, hAh^{-1}\rangle$ is attained at some $\hat{h}$.
- For any $X \in \mathfrak{k}$,

$$0 = \frac{d}{dt}\langle L_{\mathfrak{p}}, e^{tX}\hat{h}A\hat{h}^{-1}e^{-tX}\rangle|_{t=0} = \langle L_{\mathfrak{p}}, [X, \hat{h}A\hat{h}^{-1}]\rangle = \langle X, [\hat{h}A\hat{h}^{-1}, L_{\mathfrak{p}}]\rangle.$$

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- Conclusion: Time optimal trajectories on G/K coincide with the canonical geodesics.
- In low dimensions $dH(L_\mathfrak{p}) = \frac{1}{||L_\mathfrak{p}||} L_\mathfrak{p}$. Then, $L_\mathfrak{p}(t) = e^{-tQ} P e^{tQ}$ for some matrices $P \in \mathfrak{p}$ and $Q \in \mathfrak{k}$.
- In this case the extremal equation is explicitly solvable:

$$g(t) = e^{-tQ} e^{(P+Q)t}.$$

- What can we say about the general case $\frac{dL_\mathfrak{p}}{dt} = [\hat{h} A \hat{h}^{-1}, Q]$?
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