Construction of the control function for the global exact controllability and further estimates

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1-D bilinear Schrödinger equation

Let in $\mathscr{H} := L^2((0,1),\mathbb{C})$:

$$\begin{cases} i\partial_t\psi(t,x) = A\psi(t,x) + u(t)B\psi(t,x), \\ \psi(0,x) = \psi^0(x). \end{cases} \tag{BSE}$$

- $A = -\Delta$, $D(A) := H^2 \cap H_0^1((0,1),\mathbb{C}))$,
- B is a bounded symmetric operator,
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Goal: Exhibit a time and a control for the global exact controllability of (BSE).

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- Simultaneous Global Exact Controllability: Morancey and Nersesyan (2015), D. (2017).

Global Exact Controllability: Common approach

Let $\mathcal{M}$ a subspace of $\mathscr{H}$, Γ_t^u be the unitary propagator of (BSE) and $\{\phi_j\}_{j\in\mathbb{N}}$ a basis of eigenfunctions of A.

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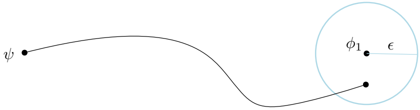

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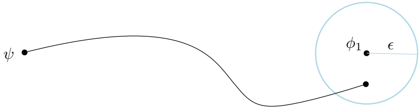

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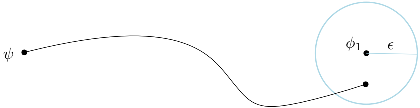

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Novelties of the work

Let $H^s_{(0)} := D(A^{\frac{s}{2}})$, $\| \cdot \|_{H^s_{(0)}} = \left(\sum_{k=1}^{\infty} |k^s \langle \cdot, \phi_k \rangle|^2 \right)^{\frac{1}{2}}$.

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1) $\forall \phi_j, \phi_k$, we construct T_n, $u_n \in L((0, T_n), \mathbb{R})$ s.t

$$\exists\ \theta \in \mathbb{R} \quad : \quad \lim_{n\to\infty} \|\Gamma^{u_n}_{T_n}\phi_j - e^{i\theta}\phi_k\|_{H^3_{(0)}} = 0.$$

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2) We exhibit r s.t. the **local exact controllability** is verified in $B_{H^3_{(0)}}(\phi_k, r)$. We also estimate $n^* \in \mathbb{N}$, $u \in L^2\big((0, \frac{4}{3\pi}), \mathbb{R}\big)$ s.t.

$$\Gamma^{u_{n^*}}_{T_{n^*}}\phi_j \in B_{H^3_{(0)}}(e^{i\theta}\phi_k, r), \qquad \Gamma^{u}_{\frac{4}{3\pi}}\Gamma^{u_{n^*}}_{T_{n^*}}\phi_j = e^{i\theta}\phi_k.$$

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3) $\exists T_1 > 0$ s.t.

$$\Gamma^{u}_{\frac{4}{3\pi}} \Gamma^{u_{n^*}}_{T_{n^*}} \Gamma^{0}_{T_1} \phi_j = \phi_k.$$

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Let $B = x^2$ be the multiplication operator.
1) *For every $\psi \in B_{H^3_{(0)}}\left(\phi_1, 1.5 \cdot 10^{-3}\right)$ there exists $u \in L^2((0, \frac{4}{3\pi}), \mathbb{R})$ s.t.*

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2) *Let $u_n(t) = n^{-1} \cos(3\pi^2 t)$. If $n = 7 \cdot 10^{63}$, then there exists $\theta \in \mathbb{R}$ s.t. for $T_n = \frac{9\pi^3 n}{8}$ it follows*

$$\left\| \Gamma^{u_n}_{T_n} \phi_2 - e^{i\theta} \phi_1 \right\|_{H^3_{(0)}} \leq 1.5 \; 10^{-3}.$$

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3) *Thus $\exists u^{lcl} \in L^2((0, \frac{4}{3\pi}), \mathbb{R})$ s.t.*

$$\Gamma^{u^{lcl}}_{\frac{4}{3\pi}} \Gamma^{u_n}_{T} \phi_2 = e^{i\theta} \phi_1.$$

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Assumptions

1. *for every* $k \in \mathbb{N}$, $\exists C_k > 0$ *s.t.* $|\langle \phi_j, B\phi_k \rangle| \geq \frac{C_k}{j^3}$, $\forall j \in \mathbb{N}$;
2. $B : D(A) \to D(A)$ *and* $Ran(B|_{H^3_{(0)}}) \subseteq H^3 \cap H^1_0$.

Sketch of the proof

1) Let $A_k(u) = \Gamma^u_T \phi_k \in H^3_{(0)}$, $F_k(v) = d_u A_k(u=0) \cdot v$. We prove the surjectivity of $F_k : X \to H^3_{(0)}$ for $X := \overline{span_{l \in \mathbb{N}} \{e^{i\pi^2 l^2 t}\}}^{L^2}$.

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4) Let $U = B(0, r) \subset X / Ker(F_k)$ s.t. $\forall u, v \in U$

$$\|H_k(u) - H_k(v)\|_{H^3_{(0)}} \leq \frac{M_1}{2} \|u - v\|_{L^2}.$$

5) F_k is an homeomorphism and $A_k : U \to A_k(U)$ too. Moreover

$$A_k(U) \supset B_{H^3_{(0)}}\Big(\phi_k, \frac{M_1}{2} r\Big).$$

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6) (Chambrion, 2012) Let ϕ_j, we compute u_n, T_n, $R_n^{j,k}$ s.t.

$$\lim_{n\to\infty} R_n^{j,k} = 0, \qquad 1 - |\langle \phi_k, \Gamma^{u_n}_{T_n} \phi_j \rangle| \le R_n^{j,k}.$$

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7) $\exists \theta \in \mathbb{R}$ s.t. $\|\Gamma^{u_n}_{T_n}\phi_j - e^{i\theta}\phi_k\|^2_{\mathscr{H}} \leq 2R_n^{j,k} + \big(R_n^{j,k}\big)^2$.

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8) By interpolation and propagation of regularity $\exists \widetilde{R}_n^{j,k}$ s.t.

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9) We estimate n^* s.t. $\Gamma^{u_{n^*}}_{T_{n^*}}\phi_j \in B_{H^3_{(0)}}\Big(e^{i\theta}\phi_k, \frac{M_1}{2} r\Big).$

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- Provide better estimates on the control functions of the local exact controllability.
- Adopt the same techniques of the work in order to establish times and controls for the simultaneous global exact controllability.

Thank you for your attention!